Quantization of a scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

Canonical momentum density

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

Impose equal-time canonical-commutation relations (=CCR)

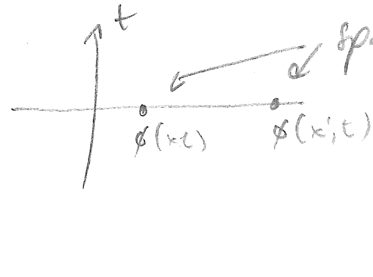
$$[\phi(\vec{x},t),\ \pi(\vec{x}',t)] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

Also

$$[\phi(\vec{x},t),\ \phi(\vec{x}',t)] = 0$$

$$[\pi(\vec{x},t),\ \pi(\vec{x}',t)] = 0$$

non trivial for $x \neq x'$



Spacelike separated fields

Recall: $[A, H] = i\frac{\partial A}{\partial t}$ Heisenberg eqns

Recall H is energy written in terms of canonical momentum.

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right]$$

$$[\phi(x,t), H] = \int d^3x' \underbrace{[\phi(x,t), \pi(x',t)]}_{i\delta} \pi(x',t)$$

$$= i\pi(x,t)$$

$$= i\frac{d\phi(x,t)}{dt}$$

problem

$$P = -\int d^3x \; \pi \vec{\nabla}\phi$$

$$[\phi(x,t), \vec{P}] = -\int d^3x' \; [\phi, \pi] \nabla\phi$$

$$= -i\nabla\phi$$

$$[\phi(x,t), P^\mu] = i\partial^\mu\phi$$

Consider a free massive scalar field. $V(\phi) = \frac{1}{2}m^2\phi^2$

Qu. fds obey same eqns as classical & $\therefore$ have same solutions

Recall from before

$$\phi(x,t) = \int \widetilde{dk} \left[a_k e^{-ik\cdot x} + a_k^\dagger e^{+ik\cdot x} \right]$$

$$\pi(x,t) = \dot\phi(x,t) = \int \widetilde{dk} \left[-i\omega a_k e^{-ik\cdot x} + i\omega a_k^\dagger e^{ik\cdot x} \right]$$

but now $a_k^* \to a_k^\dagger$

Recall also how to Invert these

$$a_k = \int d^3x\, e^{-i\vec{k}\cdot\vec{x} + i\omega t} \left[\omega\phi(x,t) + i\pi(x,t) \right] \quad \text{for any } t$$

$$a_k^\dagger = \int d^3x\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \left[\omega\phi(x,t) - i\pi(x,t) \right]$$

Then (choose same time)

$$[a_k, a_{k'}] = \int d^3x\, d^3x'\, e^{-i(\vec{k}\cdot\vec{x} + \vec{k}'\cdot\vec{x}') + i(\omega+\omega')t} \times \left(i\omega [\phi(x,t), \underbrace{\pi(x',t)]}_{i\delta(x-x')} + i\omega' [\pi(x,t), \underbrace{\phi(x',t)]}_{-i\delta(x'-x)} \right) = 0$$

$[a_k^\dagger, a_{k'}^\dagger] = 0$ for similar reason

$$[a_k, a_{k'}^\dagger] = \int d^3x\, d^3x'\, e^{-i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}') + i(\omega-\omega')t} \left(-i\omega [\phi(x,t), \underbrace{\pi(x',t)]}_{i\delta(x-x')} + i\omega' [\pi(x,t), \underbrace{\phi(x',t)]}_{-i\delta(x'-x)} \right)$$

$$= e^{i(\omega-\omega')t} (\omega+\omega') \underbrace{\int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} \Rightarrow \omega = \omega'$$

$$= (2\pi)^3 (2\omega)\, \delta^{(3)}(\vec{k}-\vec{k}')$$

choose $t=t'$

$$[a_k, a_{k'}] = \int d^3x \; d^3x' \; e^{-i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{x}') + i(\omega+\omega')t}$$

$$\left\{ i\omega [\phi(x,t), \underbrace{\pi(x',t)}_{i\delta(x-x')}] + i\omega' [\underbrace{\pi(x,t), \phi(x',t)}_{-i\delta(x-x')}] \right\}$$

$$= \int d^3x \; e^{-i(\vec{k}+\vec{k}')\cdot\vec{x} + i(\omega+\omega')t} \, [\omega' - \omega]$$

$$= (\omega'-\omega) e^{i(\omega+\omega')t} \underbrace{\int d^3x \, e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta(\vec{k}+\vec{k}')}$$

$$\Downarrow$$

$$\vec{k}' = -\vec{k} \Rightarrow \omega' = \omega$$

$$= 0$$

$$[a^\dagger_k, a^\dagger_{k'}] = 0 \quad \text{Also}$$

$$[a_k, a^\dagger_{k'}] = \int d^3x \, d^3x' \; e^{-i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}') + i(\omega-\omega')t}$$

$$\left\{ -i\omega [\underbrace{\phi, \pi}_{i\delta(x-x')}] + i\omega' [\underbrace{\pi, \phi}_{-i\delta(x-x')}] \right\}$$

$$= \int d^3x \; e^{-i(\vec{k}-\vec{k}')\cdot\vec{x} + i(\omega-\omega')t} \, [\omega+\omega']$$

$$= (2\pi)^3 \delta(\vec{k}-\vec{k}') \, e^{i(\omega-\omega')t} \, [\omega+\omega']$$

$$\Downarrow$$

$$\vec{k}' = \vec{k} \Rightarrow \omega' = \omega$$

$$= (2\pi)^3 (2\omega) \, \delta(\vec{k}-\vec{k}')$$

- skip

Define $\phi(x) = \phi^+(x) + \phi^-(x)$ → may not need these

$$\phi^+(x) = \int \widetilde{dk}\; a_k e^{-ik\cdot x}$$

$$\phi^-(x) = \int \widetilde{dk}\; a_k^\dagger e^{ik\cdot x}$$

$[\phi^+(x), \phi^+(x')] = 0$ because $[a_k, a_{k'}] = 0$

$[\phi^-(x), \phi^-(x')] = 0$ because $[a_k^\dagger, a_{k'}^\dagger] = 0$

Define $\Delta_+(x-x') = [\phi^+(x), \phi^-(x')]$

$$= \int \widetilde{dk}\,\widetilde{dk}'\; [a_k, a_{k'}^\dagger]\, e^{-ik\cdot x + ik'\cdot x'}$$

$$= \int \widetilde{dk}\; \frac{d^3k'}{(2\pi)^3(2\omega)}\; (2\pi)^3(2\omega)\,\delta(\vec{k}-\vec{k}')\; e^{-ik\cdot(x-x')}$$

$\Downarrow$

$\delta(\omega-\omega')$

$$= \int \widetilde{dk}\; e^{-ik\cdot(x-x')}$$

$$= \int \frac{d^3k}{(2\pi)^3(2\omega)}\; e^{-ik\cdot(x-x')}$$

Indeed, a fcn of $x-x'$

Because of this, answer is not $\delta(\vec{x}-\vec{x}')$

$$[\phi^-(x), \phi^+(x')] = -[\phi^+(x'), \phi^-(x)] = -\Delta_+(x'-x)$$

Now use $[a,a]$, $[a,a^\dagger]$ to

Compute unequal time commutators (a.k.a. the invariant Δ-fcn)

$$\Delta(\vec{x},t) = [\phi(\vec{x},t),\ \phi(0,0)]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(e^{-ik\cdot x} \underbrace{[a_k, a_{k'}^\dagger]}_{(2\pi)^3\, 2\omega\, \delta^{(3)}(\vec{k}-\vec{k}')} + e^{ik\cdot x} \underbrace{[a_k^\dagger, a_{k'}]}_{-(2\pi)^3\, 2\omega\, \delta^{(3)}(\vec{k}'-\vec{k})} \Big)$$

$$\Downarrow$$

$$\omega' = \omega$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \left(e^{-ik\cdot x} - e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^3 (2\omega)} \big(\underbrace{e^{i\vec{k}\cdot\vec{x} - i\omega t} - e^{-i\vec{k}\cdot\vec{x} + i\omega t}}_{2i\sin(\vec{k}\cdot\vec{x} - \omega t)} \big)$$

Check that this vanishes for equal time commutator $(t=0)$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \left[e^{+i\vec{k}\cdot\vec{x}} - e^{-i\vec{k}\cdot\vec{x}} \right] = 2i\sin(\vec{k}\cdot\vec{x})$$

$\uparrow$ odd function

Let $k_x' = -k_x$. observe that
$$\int_{k_x=-\infty}^{k_x=\infty} dk_x\, f(k_x) = \int_{k_x'=\infty}^{k_x'=-\infty} (-dk_x')\, f(-k_x')$$

$$= \int_{k_x'=-\infty}^{k_x'=\infty} dk_x'\, f(-k_x') = \int_{k_x=-\infty}^{k_x=\infty} dk_x\, f(-k_x)$$

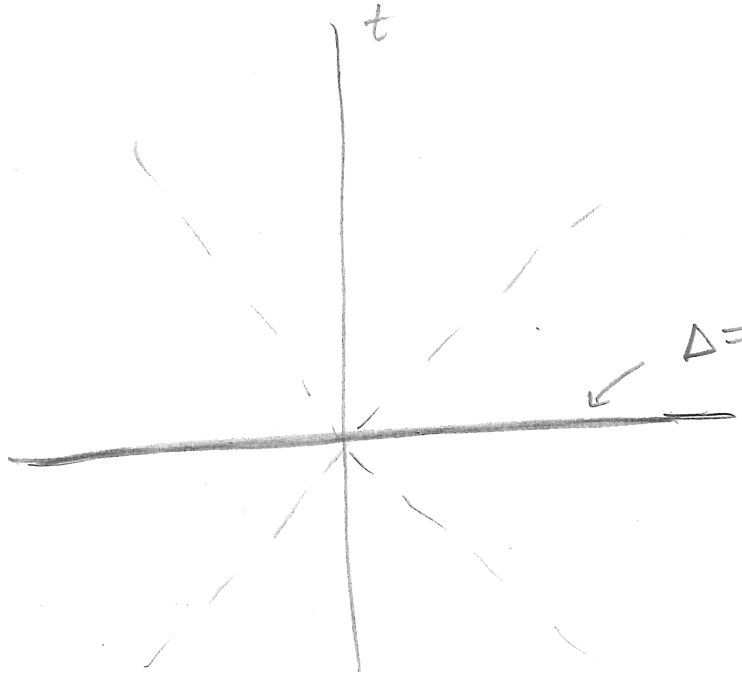
Therefore can let $\vec{k} \to -\vec{k}$ in 2nd term above

$$\int \frac{d^3k}{(2\pi)^3 2\omega} \left[e^{i\vec{k}\cdot\vec{x}} - e^{i\vec{k}\cdot\vec{x}} \right] = 0 \quad \checkmark$$

but not if $t \neq 0 \ldots$

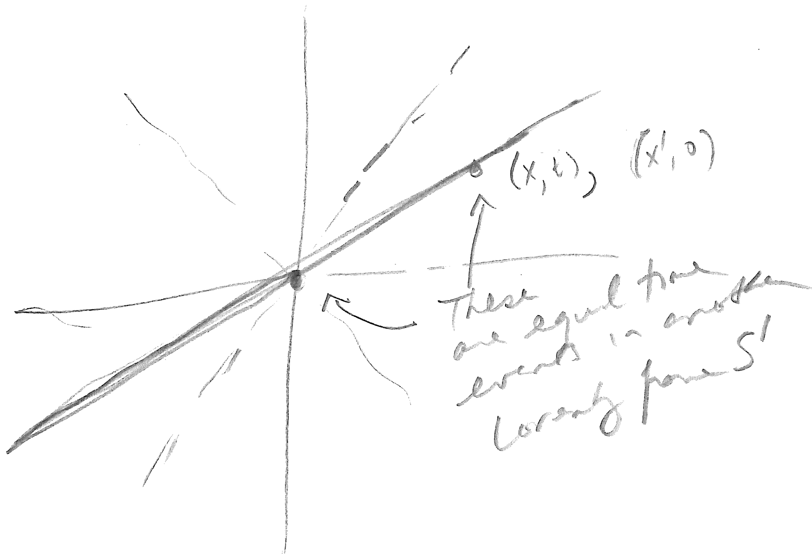
$$\Delta(\vec{x},t) = [\phi(\vec{x},t), \phi(0,0)] = \int d\tilde{k}\,[e^{-ik\cdot x} - e^{ik\cdot x}]$$

We just showed that $\Delta(\vec{x},0) = 0$



This makes sense because measurements of $\phi(x,0)$ + $\phi(0,0)$ shouldn't affect each other.

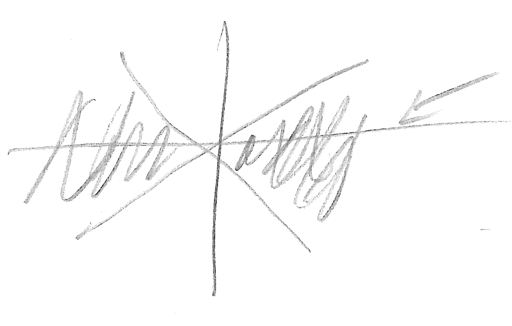
But measurements at any space like separated intervals shouldn't affect each other.



Observe that $\Delta(x)$ is Lorentz invariant

$$= \int d\tilde{k}'\,[e^{-ik'\cdot x'} - e^{ik'\cdot x'}]$$

$= 0$ for $t' = 0$



$\Delta(x) = 0$ in spacelike separated region

But not outside this region

Consider (Problem)

$$[\pi(x,t), \phi(0,0)] = \left[\frac{d\phi}{dt}(x,t), \phi(0,0)\right] = \frac{d}{dt}\Delta(x,t)$$

$$= \frac{d}{dt}\int \frac{d^3k}{(2\pi)^3(2\omega)}\left[e^{-i\omega t + i\vec{k}\cdot\vec{x}} - e^{i\omega t - i\vec{k}\cdot\vec{x}}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3}\left[-\frac{i}{2}e^{-i\omega t + i\vec{k}\cdot\vec{x}} - \frac{i}{2}e^{-i\omega t - i\vec{k}\cdot\vec{x}}\right]$$

Again, check equal time commutators

$$[\pi(x,0), \phi(0,0)] = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left[e^{ikx} + e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$= -\frac{i}{2}\left[\delta(\vec{x}) + \delta(\cancel{-}\vec{x})\right]$$

$$= -i\delta(\vec{x})$$

↑ vanishes except at $\vec{x}=0$

agrees w/ equal time commutator

Finally $[\pi(x,t), \pi(0,0)]$

Can show that $[\pi(x,0), \pi(0,0)] = 0$

problem

Recall energy density

$$T^{00} = \frac{1}{2}\left[m^2 \phi^2 + \dot{\phi}^2 + (\nabla\phi)^2 \right]$$

$$k \cdot x = \omega t - \vec{k}\cdot\vec{x}$$
$$\omega = \sqrt{k^2 + m^2}$$

$$\phi(x,t) = \int \tilde{dk} \left[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right]$$

$$\dot{\phi}(x,t) = \int \tilde{dk} \left[-i\omega a_k e^{-ik\cdot x} + i\omega a_k^\dagger e^{ik\cdot x} \right]$$

$$\vec{\nabla}\phi(x,t) = \int \tilde{dk} \left[i\vec{k} a_k e^{-ik\cdot x} - i\vec{k} a_k^\dagger e^{ik\cdot x} \right]$$

Hamiltonian (energy)

$$H = \int d^3x \, T^{00}$$

$$= \frac{1}{2} \int d^3x \, \tilde{dk} \, \tilde{dk'} \Big[a_k a_{k'} e^{-i(k+k')\cdot x} \left(m^2 - \omega\omega' - \vec{k}\cdot\vec{k}' \right)$$
$$+ a_k^\dagger a_{k'}^\dagger e^{i(k+k')\cdot x} \left(m^2 - \omega\omega' - \vec{k}\cdot\vec{k}' \right)$$
$$+ a_k a_{k'}^\dagger e^{-i(k-k')\cdot x} \left(m^2 + \omega\omega' + \vec{k}\cdot\vec{k}' \right)$$
$$+ a_k^\dagger a_{k'} e^{i(k-k')\cdot x} \left(m^2 + \omega\omega' + \vec{k}\cdot\vec{k}' \right) \Big]$$

$$\int d^3x\, e^{-i(k+k')\cdot x} = e^{-i(\omega+\omega')t} \underbrace{\int d^3x\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\,\delta(\vec{k}+\vec{k}')}$$

$$\Rightarrow \vec{k}' = -\vec{k} \Rightarrow \omega' = \omega$$

$$= e^{-2i\omega t}\,(2\pi)^3\,\delta(\vec{k}+\vec{k}')$$

$$\int d^3x\, e^{-i(k-k')\cdot x} = e^{-i(\omega-\omega')t} \underbrace{\int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')}$$

$$\Rightarrow \vec{k}' = \vec{k} \Rightarrow \omega' = \omega$$

$$= (2\pi)^3\,\delta(\vec{k}-\vec{k}')$$

$$H = \frac{1}{2}\int d\tilde{k}\,\frac{d^3\vec{k}'}{(2\omega)}\Big[a_k a_{k'}\, e^{-2i\omega t}\,\delta(\vec{k}+\vec{k}')\,(m^2-\omega^2+\vec{k}^2)$$

$$+ a_k^\dagger a_{k'}^\dagger\, e^{2i\omega t}\,\delta(\vec{k}+\vec{k}')\,(m^2-\omega^2+\vec{k}^2)$$

$$+ a_k a_{k'}^\dagger\,\delta(\vec{k}-\vec{k}')\,(m^2+\omega^2+\vec{k}^2)$$

$$+ a_k^\dagger a_{k'}\,\delta(\vec{k}-\vec{k}')\,(m^2+\omega^2+\vec{k}^2)\Big]$$

since $\omega' = \omega$

Now $\omega^2 = \vec{k}^2 + m^2$ $\Rightarrow$ 1st two lines vanish

(good thing, because time dependent + E should be conserved)

$$H = \frac{1}{2}\int d\tilde{k}\ \frac{1}{2\omega}\left[a_k a_k^\dagger\ 2\omega^2 + a_k^\dagger a_k\ 2\omega^2\right]$$

$$= \int d\tilde{k}\ \frac{\omega}{2}\left[a_k a_k^\dagger + a_k^\dagger a_k\right]$$

(Take a deep breath)

Recall for harmonic oscillator

$$\frac{\omega}{2}(a^\dagger a + a a^\dagger) = \omega a^\dagger a + \frac{1}{2}\omega$$

zero point energy (energy of ground st)

$$H = \int d\tilde{k}\left(\omega a_k^\dagger a_k + \frac{\omega}{2}[a_k, a_k^\dagger]\right)$$

Recall $[a_k, a_{k'}^\dagger] = (2\pi)^3(2\omega)\,\delta(\vec{k}-\vec{k}') \longrightarrow \infty$ for $k = k'$!

Let's try to understand this infinity

$$= (2\omega)\int d^3x\ e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$= (2\omega)\int d^3x\ 1 \qquad = \text{volume of space} = V$$

$$H = \int d\tilde{k}\left(\omega a_k^\dagger a_k + \omega^2 V\right)$$

$|0\rangle$ = ground state of scalar field theory

defined by $\quad a_k|0\rangle = 0 \quad$ for all $\vec{k}$

Energy of ground state

$$H|0\rangle = V\int d\tilde{k}\,\omega^2$$

$$= V\int \frac{d^3k}{(2\pi)^3}\frac{\omega}{2} = V\int_0^\infty \frac{4\pi k^2\,dk\,\sqrt{k^2+m^2}}{(2\pi)^3\,2}$$

$$\sim \boxed{\frac{V}{(2\pi)^3}\frac{\pi}{2}k_{max}^4}$$

also ∞!

↓

ultraviolet divergence

Unless there is a cut-off on the momenta of the scalar field ie a minimum wavelength

Is space granular?

Ad hoc solution: overall level of energy is unobservable (except in gravity?)

↓

ignore it

$$H|0\rangle = 0$$

$$H = \int d\tilde{k}\,(\omega a_k^\dagger a_k)$$

[HW problem]

$$T^{0i} = -\dot{\phi}\,\partial_i \phi$$

Some ambiguity because could write also $-(\vec{\nabla}\phi)\dot{\phi}$ but $\vec{\nabla}[\phi,\pi]$ vanishes in integral

$$\vec{P} = \int d^3x \left[-\dot{\phi}\,\vec{\nabla}\phi \right]$$

$$= \int d\tilde{k}\, d\tilde{k}'\, d^3x \Big[a_k a_{k'} e^{-i(k+k')\cdot x} (-\omega \vec{k}')$$
$$a_k^\dagger a_{k'}^\dagger e^{i(k+k')\cdot x} (-\omega \vec{k}')$$
$$a_k a_{k'}^\dagger e^{-i(k-k')\cdot x} (\omega \vec{k}')$$
$$a_k^\dagger a_{k'} e^{i(k-k')\cdot x} (\omega \vec{k}') \Big]$$

$$= \int d\tilde{k} \frac{d^3k'}{(2\omega)} \Big[a_k a_{k'} e^{-2i\omega t} \delta(\vec{k}+\vec{k}')(-\omega \vec{k}')$$
$$+ a_k^\dagger a_{k'}^\dagger e^{2i\omega t} \delta(\vec{k}+\vec{k}')(-\omega \vec{k}')$$
$$+ a_k a_{k'}^\dagger \delta(\vec{k}-\vec{k}')(\omega \vec{k}')$$
$$+ a_k^\dagger a_{k'} \delta(\vec{k}-\vec{k}')(\omega \vec{k}) \Big]$$

$$= \frac{1}{2} \int d\tilde{k} \Big[a_k a_{-k} e^{-2i\omega t} \vec{k} + a_k^\dagger a_{-k}^\dagger e^{2i\omega t} \vec{k}$$
$$+ a_k a_k^\dagger \vec{k} + a_k^\dagger a_k \vec{k} \Big]$$

Recall $\int d\tilde{k} = \int d\tilde{k}'$ where $k' = -k$

$$\int d\tilde{k}\; a_k a_{-k}\, e^{-2i\omega t}\, \vec{k} = \int d\tilde{k}'\; \underbrace{a_{-k'} a_{k'}}_{\text{commute}}\, e^{-2i\omega t}(-\vec{k}')$$

$$=$$

$$= \int d\tilde{k}'\; a_{k'} a_{-k'}\, e^{-2i\omega t}(-\vec{k}')$$

$$= 0$$

$$\vec{P} = \frac{1}{2}\int d\tilde{k}\, [\underbrace{a_k a_k^\dagger}_{a_k^\dagger a_k + \underbrace{[a_k, a_k^\dagger]}_{2\omega V}} + a_k^\dagger a_k]\, \vec{k}$$

$$= \int d\tilde{k}\; a_k^\dagger a_k \vec{k} + \frac{V}{2}\underbrace{\int \frac{d^3k}{(2\pi)^3}\vec{k}}_{0}$$

$$\boxed{P^\mu = \int d\tilde{k}\; a_k^\dagger a_k\, k^\mu}$$

Define $|\vec{k}\rangle = a_k^\dagger |0\rangle$

$$H|\vec{k}\rangle = \left[\int d\tilde{k}'\, \omega'\, a_{k'}^\dagger a_{k'}\right] a_k^\dagger |0\rangle$$

$$= \int \frac{d^3k'}{(2\pi)^3 (2\omega')}\, \omega'\, a_{k'}^\dagger \Big(\underbrace{[a_{k'}, a_k^\dagger]}_{(2\omega)(2\pi)^3 \delta(\vec{k}'-\vec{k})} + a_k^\dagger a_{k'} \Big) |0\rangle$$

annihilate

$\hookrightarrow \omega = \omega'$

$$= \omega\, a_k^\dagger |0\rangle$$

$$= \omega |\vec{k}\rangle$$

so $|\vec{k}\rangle$ has energy ω ie $\hbar\omega = k\sqrt{k^2+m^2}$

You will calculate $\vec{P}$, the total momentum of the theory, and will find that $\vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$

ie $|\vec{k}\rangle$ has momentum $\vec{k}$ ie $\hbar\vec{k}$

$|\vec{k}\rangle$ represents a <u>particle</u> w/ momentum $\hbar\vec{k}$ and energy $\hbar\omega$

+ ∴ mass $\sqrt{E^2-p^2} = \sqrt{\omega^2-\vec{k}^2} = m$ ✓

quantize a field, get a particle

But not just one!

$$|\vec{k},\vec{k}'\rangle = a^\dagger_k a^\dagger_{k'} |0\rangle$$

$\int d\vec{k}''\, \omega''_k\, a^\dagger_{k''} \underbrace{a_{k''}\, a^\dagger_k}_{[a_{k''}, a^\dagger_k] + a^\dagger_k a_{k''}} a^\dagger_{k'} |0\rangle$ ← then work the all out to give

$$H\, |\vec{k},\vec{k}'\rangle = (\omega + \omega')\, |\vec{k},\vec{k}'\rangle$$

$$\vec{P}\, |\vec{k},\vec{k}'\rangle = (\vec{k} + \vec{k}')\, |\vec{k},\vec{k}'\rangle$$

A state of energy $\hbar\omega + \hbar\omega'$

and momentum $\hbar\vec{k} + \hbar\vec{k}'$

ie 2 particles. If $\vec{k} = \vec{k}'$ then 2 ptcls of same quantum. ↓ bosons

in fact, can create an ∞ # of particles

$a^\dagger$ = creation operator; ~~also $a_k|\vec{k}\rangle = |0\rangle$~~

$|0\rangle$ represents state of no particles, ie the <u>vacuum</u>

(→ vacuum has ∞ energy!)

⇓

cosmological constant?

$$a_k|\vec{k}\rangle = a_k\,(a^\dagger_k|0\rangle) = [a_k, a^\dagger_k]|0\rangle \sim |0\rangle$$

a_k is an annihilation operator

Explain: these are bosons.

~~$a^\dagger$~~ ① can put more than 1 in single state

$(a_k^\dagger)^2 |0\rangle$

② $a_{k}^\dagger a_{k'}^\dagger |0\rangle = a_{k'}^\dagger a_k^\dagger |0\rangle$ → so symmetric under interchange.

$a^\dagger$ commute w/ each other

QM $\begin{cases} a^\dagger & \text{raising} \\ a & \text{lowering} \end{cases}$ QFT $\begin{cases} a_k^\dagger & \text{creation} \\ a_k & \text{annihilate} \end{cases}$

Two real scalar fields

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - V(\phi_1, \phi_2)$$

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \dot{\phi}_i \qquad i = 1, 2$$

equal time comm relns

$$[\phi_i(x,t), \phi_j(x',t)] = 0$$
$$[\pi_i(x,t), \pi_j(x',t)] = 0$$
$$[\phi_i(x,t), \pi_j(x',t)] = i\,\delta_{ij}\,\delta(x-x')$$

Free massive fields

$$V(\phi_1, \phi_2) = \frac{1}{2} m_1^2 \phi_1^2 + \frac{1}{2} m_2^2 \phi_2^2$$

$$E\text{-}L \Rightarrow (\partial^2 + m_i^2)\phi_i = 0$$

$$\text{soln} \Rightarrow \phi_i(x) = \int d\vec{k}\, \left[a_{ik} e^{-ik\cdot x} + a^\dagger_{ik} e^{ik\cdot x} \right] \Big|_{k^2 = m_i^2}$$

$$[a_{ik}, a_{jk}] = 0$$
$$[a^\dagger_{ik}, a^\dagger_{jk}] = 0$$

$$[a_{ik}, a^\dagger_{jk}] = (2\pi)^3 (2\omega)\, \delta_{ij}\, \delta(\vec{k} - \vec{k}')$$

$$H = \sum_{i=1}^{2} \int d\vec{k}\; \omega_i\, a^\dagger_{ik} a_{ik} \qquad \omega_i = \sqrt{k^2 + m_i^2}$$

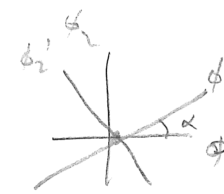
Suppose $V(\phi_1, \phi_2) = V(\phi_1^2 + \phi_2^2)$

Example: 2 free fds w/ $\underline{\text{equal}}$ mass $\quad V = \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2)$

$\phi' = \phi_1' + i\phi_2'$
$= \phi_1 \cos\alpha + \phi_2 \sin\alpha + i\phi_2 \cos\alpha$
$- i\phi_1 \sin\alpha$
$= (\phi_1 + i\phi_2) e^{-i\alpha}$

$\mathcal{L}$ is invariant under rotations in the $\phi_1 \phi_2$ plane

$$\begin{pmatrix} \phi_1(\alpha) \\ \phi_2(\alpha) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$



Then

$$\mathcal{L}_\alpha = \mathcal{L}(\phi_1(\alpha), \phi_2(\alpha)) = \mathcal{L}(\phi_1, \phi_2)$$

$$0 = \left.\frac{d\mathcal{L}_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{\partial \mathcal{L}}{\partial \phi_1} \frac{\partial \phi_1}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \frac{\partial(\partial_\mu \phi_1)}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \phi_2} \frac{\partial \phi_2}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \frac{\partial(\partial_\mu \phi_2)}{\partial \alpha}$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \right) \phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \partial_\mu \phi_2 + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \right)(-\phi_1) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} (-\partial_\mu \phi_1)$$

$$= \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \phi_2 - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \phi_1 \right]}_{j^\mu}$$

$j^\mu \leftarrow$ conserved current associated w/ invariance under rotations in $\phi_1 \phi_2$ plane

$j^\mu = (\partial^\mu \phi_1) \phi_2 - (\partial^\mu \phi_2) \phi_1$ $\qquad$ (Noether)

$$Q = \int j^0 \, d^3x$$

OK because $[\phi_i, \pi_j] \sim \delta_{ij}$

$$= \int d^3x \, (\pi_1 \phi_2 - \pi_2 \phi_1) = \int d^3x \, (\pi_1 \phi_2 - \phi_1 \pi_2)$$

[HW problem]

$$= \int d^3x \, d\tilde{k} \, d\tilde{k}' \Big(e^{-i(k+k')\cdot x} \left[-i\omega \, a_{1k} a_{2k'} + i\omega' a_{1k} a_{2k'} \right]$$

$$+ e^{i(k+k')\cdot x} \left[+i\omega \, a^\dagger_{1k} a^\dagger_{2k'} - i\omega' a^\dagger_{1k} a^\dagger_{2k'} \right]$$

$$+ e^{-i(k-k')\cdot x} \left[-i\omega \, a_{1k} a^\dagger_{2k'} - i\omega' a_{1k} a^\dagger_{2k'} \right]$$

$$+ e^{i(k-k')\cdot x} \left[i\omega \, a^\dagger_{1k} a_{2k'} + i\omega' a^\dagger_{1k} a_{2k'} \right] \Big)$$

$$= \int d\tilde{k} \, \frac{d^3k'}{(2\omega)} \left[0 + 0 + \delta(\vec{k}-\vec{k}') \left\{ (-2i\omega) a_{1k} a^\dagger_{2k} + (2i\omega) a^\dagger_{1k} a_{2k} \right\} \right]$$

$$= \int d\tilde{k} \; i\left(a^\dagger_{1k} a_{2k} - a_{1k} a^\dagger_{2k} \right)$$

$$= \int d\tilde{k} \; i\left(a^\dagger_{1k} a_{2k} - a^\dagger_{2k} a_{1k} \right)$$

no problem w/ "zero pt charge"

2 real scalar fields
= Complex scalar fields

Let $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$\phi \to \phi' = \phi e^{-i\alpha}$

Then $|\phi|^2 = \phi^* \phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$

Also $|\partial\phi|^2 = (\partial_\mu \phi^*)(\partial^\mu \phi) = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2$

$$\mathcal{L} = |\partial\phi|^2 - V(|\phi|^2)$$ ← provided there is rotation symmetry in $\phi_1 \phi_2$ plane

$$= \dot{\phi}^* \dot{\phi} - (\nabla\phi^*)\cdot(\nabla\phi) - V(|\phi|^2)$$

For a free complex field $V(|\phi|^2) = m^2 |\phi|^2$

Canonical momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* = \frac{1}{\sqrt{2}}(\dot{\phi}_1 - i\dot{\phi}_2) = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)$$

Equal time Comm rels

$$[\phi, \phi] = [\phi, \phi^\dagger] = [\phi^\dagger, \phi^\dagger] = 0$$ Similarly for $\pi, \pi^\dagger$

$$[\phi(x,t), \pi(x',t)] = \frac{1}{2}[\phi_1 + i\phi_2, \pi_1 - i\pi_2]$$

$$= \frac{1}{2}[\phi_1, \pi_1] + \frac{1}{2}[\phi_2, \pi_2] = i\delta(x-x')$$

$$[\phi(x,t), \pi^\dagger(x',t)] = \frac{1}{2}[\phi_1 + i\phi_2, \pi_1 + i\pi_2] = 0$$

$$[\phi^\dagger(x,t), \pi^\dagger(x',t)] = i\delta(x-x')$$

HW: $T^{\mu\nu}, \partial^\mu$

Complex scalar field

$$\mathcal{L} = (\partial\phi^*)(\partial\phi) - V$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right)$$

$$\boxed{\delta\mathcal{L} = \partial_\mu(\partial^\mu\phi^*\,\delta\phi) + \partial_\mu(\partial^\mu\phi\,\delta\phi^*)}$$

spacetime translation: $\delta\phi = a_\nu\partial^\nu\phi$, $\delta\mathcal{L} = a_\nu\partial^\nu\mathcal{L} = a_\nu\partial_\mu(\eta^{\mu\nu}\mathcal{L})$

$$0 = a_\nu\partial_\mu\left[\partial^\mu\phi^*\partial^\nu\phi + \partial^\mu\phi\,\partial^\nu\phi^* - \eta^{\mu\nu}\mathcal{L}\right]$$

$$\boxed{T^{\mu\nu} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\mu\phi\,\partial^\nu\phi^* - \eta^{\mu\nu}\mathcal{L}}$$

[check: $\frac{1}{2}\partial^\mu(\phi_1 - i\phi_2)\partial^\nu(\phi_1 + i\phi_2) + \frac{1}{2}\partial^\mu(\phi_1 + i\phi_2)\partial^\nu(\phi_1 - i\phi_2) - \eta^{\mu\nu}\mathcal{L}$

$= \partial^\mu\phi_1\partial^\nu\phi_1 + \partial^\mu\phi_2\partial^\nu\phi_2 - \eta^{\mu\nu}\mathcal{L}$)

U(1) phase rotation

$$\phi \to e^{-i\alpha}\phi$$

[check $\phi_1 + i\phi_2 \to (\cos\alpha - i\sin\alpha)(\phi_1 + i\phi_2) = \phi_1\cos\alpha + \phi_2\sin\alpha + i(\phi_2\cos\alpha - \phi_1\sin\alpha)$]

$\delta\phi = -i\alpha\phi$, $\delta\phi^* = i\alpha\phi^*$

$\delta\mathcal{L} = -i\alpha\,\partial_\mu[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*]$

$$\boxed{J^\mu = i(\phi^*\partial^\mu\phi - \phi(\partial^\mu\phi^*))}$$

[check: $\frac{1}{2}i[(\phi_1 - i\phi_2)\partial(\phi_1 + i\phi_2) - (\phi_1 + i\phi_2)\partial(\phi_1 - i\phi_2)]$
$= \phi_2\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2$]

For a free complex scalar field $\left(\Rightarrow (\partial^2 + m^2)\phi = 0 \right)$

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

no longer hermitian, conjugates

$$= \frac{1}{\sqrt{2}} \int \widetilde{dk} \left[(a_{1k} + i a_{2k}) e^{-ik\cdot x} + (a^\dagger_{1k} + i a^\dagger_{2k}) e^{ik\cdot x} \right]$$

$$\equiv \int \widetilde{dk} \left[a_k e^{-ik\cdot x} + b^\dagger_k e^{ik\cdot x} \right]$$

where $\begin{cases} a_k \equiv \frac{1}{\sqrt{2}}(a_{1k} + i a_{2k}) \\ b_k \equiv \frac{1}{\sqrt{2}}(a_{1k} - i a_{2k}) \end{cases}$ $\quad \left[\begin{aligned} a_{1k} &= \frac{1}{\sqrt{2}}(a_k + b_k) \\ a_{2k} &= \frac{1}{\sqrt{2}i}(a_k - b_k) \end{aligned} \right]$

[could then show $a_k = \int d^3x\, e^{ik\cdot x}\, i\overleftrightarrow{\partial_0}\, \phi = \int d^3x\, e^{ik\cdot x}[\omega\phi + i\pi^\dagger]$

$b_k = \int d^3x\, e^{ik\cdot x}\, i\overleftrightarrow{\partial_0}\, \phi^* = \int d^3x\, e^{ik\cdot x}[\omega\phi^* + i\pi]$]

HW compute all comm of $a, a^\dagger, b, b^\dagger$

$$[a_k, a^\dagger_{k'}] = \frac{1}{2}[a_{1k}, a^\dagger_{1k'}] + \frac{1}{2}[a_{2k}, a^\dagger_{2k'}] = (2\pi)^3 (2\omega)\, \delta^{(3)}(\vec{k} - \vec{k}')$$

$$[b_k, b^\dagger_{k'}] = \quad = (2\pi)^3 (2\omega)\, \delta^{(3)}(\vec{k} - \vec{k}')$$

All others vanish $\left. \begin{aligned} [a, a] = [a, b] = [b, b] = 0 \\ [a^\dagger, a^\dagger] = [a^\dagger, b^\dagger] = [b^\dagger, b^\dagger] = 0 \end{aligned} \right\}$ obvious

$$[a, b^\dagger] = \frac{1}{2}[a_{1k}, a^\dagger_{1k}] - \frac{1}{2}[a_{2k}, a^\dagger_{2k}] = 0$$

$$[a^\dagger, b] = 0$$

$$\boxed{\begin{matrix} a & a^\dagger \\ b & b^\dagger \end{matrix}}$$

$$H = \int d\tilde{k}\ \omega(\underbrace{a_{1k}^\dagger a_{1k}} + a_{2k}^\dagger a_{2k})$$

$$\frac{1}{2}(a_k^\dagger + b_k^\dagger)(a_k + b_k) + \frac{1}{2}(a_k^\dagger - b_k^\dagger)(a_k - b_k)$$

$$= a_k^\dagger a_k + b_k^\dagger b_k$$

$$\vec{P} = \int d\tilde{k}\ \vec{k}(a_k^\dagger a_k + b_k^\dagger b_k)$$

vacuum: $a_k|0\rangle = b_k|0\rangle = 0$

You will show ~~(HW)~~ (HW)

$$Q = \int d\tilde{k}\ i(a_{1k}^\dagger a_{2k} - a_{2k}^\dagger a_{1k}) \rightarrow \cancel{a_k^\dagger a_k - b_k^\dagger b_k}$$

$$= \int d\tilde{k}\ i\left(\frac{1}{2i}[a_k^\dagger + b_k^\dagger][a_k - b_k] + \frac{1}{2i}[a_k^\dagger - b_k^\dagger](a_k + b_k)\right)$$

$a_k a_k^\dagger - b_k b_k^\dagger$

$$\boxed{Q = \int d\tilde{k}\,[a_k^\dagger a_k - b_k^\dagger b_k]}$$

$\rightarrow$ HW ~~[illegible]~~

so $a_k^\dagger|0\rangle$ is a state w/ $\vec{k}$, E + charge 1

$b_k^\dagger|0\rangle$ is a state w/ $\vec{k}$, E + charge -1

particle + antiparticle $\rightarrow$ $b_k^\dagger$ creates the antiparticle

Real scalar

$$\phi(x) = \int d\tilde{k} \left[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right]$$

$$\pi = \dot\phi = \int d\tilde{k} \left[-i\omega a_k e^{-ikx} + i\omega a_k^\dagger e^{ik\cdot x} \right]$$

$$[\phi, \pi]_= = \int d\tilde{k}\, d\tilde{k}' \Big[i\omega' \underbrace{[a_k, a_{k'}^\dagger]}_{(2\pi)^3 (2\omega)\,\delta(\;)} e^{\cdots} - i\omega' \underbrace{[a_k^\dagger a_{k'}]}_{-(2\pi)^3(2\omega)\,\delta(\;)} e^{\cdots} \Big]$$

$$= i\,\delta(x - x')$$

Complex scalar

$$\phi(x) = \int d\tilde{k} \left[a_k e^{-ik\cdot x} + b_k^\dagger e^{ikx} \right]$$

$$\phi^*(x) = \int d\tilde{k} \left[a_k^\dagger e^{ik\cdot x} + b_k e^{-ik\cdot x} \right]$$

$$\pi = \dot\phi^* = \int d\tilde{k} \left[i\omega a_k^\dagger e^{ikx} - i\omega b_k e^{-ikx} \right]$$

$$[\phi, \pi] = \int d\tilde{k}\, d\tilde{k}' \Big[i\omega' \underbrace{[a_k, a_{k'}^\dagger]}_{(2\pi)^3 (2\omega)\,\delta} e^{\cdots} - i\omega' \underbrace{[b_k^\dagger b_{k'}]}_{-(2\pi)^3 (2\omega)\,\delta(\;)} e^{\cdots} \Big]$$

choose $b_k^\dagger$ instead of b so that get a minus sign to compensate sign in $\underline{(-i\omega)}$