Scalar fields

Recall wave eqn from Phys 3000

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \qquad \text{(wave eqn w/ wave speed = c)}$$

Set $c = 1$.

Modify eqn by adding a term w/ no derivatives (a "mass" term)

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0 \qquad \text{(Klein Gordon eqn)}$$

Ansatz:

$$\phi = e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

Obeys Klein Gordon eqn if

$$-\omega^2 + \vec{k}^2 + m^2 = 0$$

$\Rightarrow \vec{k}$ is arbitrary, $\omega = \pm\sqrt{k^2 + m^2}$

For $m = 0$, $\omega = |\vec{k}|$ as expected for wave eqn

Henceforth we will always choose the positive root $\omega \equiv \sqrt{\vec{k}^2 + m^2}$ and write the other solution seperately

$$\phi = A_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + B_{\vec{k}}\, e^{i\omega t + i\vec{k}\cdot\vec{x}}$$

Most general solution

$$\phi = \int d^3\vec{k}\left[A_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + B_{\vec{k}}\, e^{i\omega t + i\vec{k}\cdot\vec{x}} \right]\Bigg|_{\omega = \sqrt{\vec{k}^2 + m^2}}$$

↑ implicit from now on

Consider complex conjugate

$$\phi^* = \int d^3\vec{k}\left[A^*_{\vec{k}}\, e^{i\omega t - i\vec{k}\cdot\vec{x}} + B^*_{\vec{k}}\, e^{-i\omega t - i\vec{k}\cdot\vec{x}} \right]$$

Let $\vec{k} = -\vec{k}'$

$$= \int d^3\vec{k}' \left[A^*_{-\vec{k}'}\, e^{+i\omega t + i\vec{k}'\cdot\vec{x}} + B^*_{-\vec{k}'}\, e^{-i\omega t + i\vec{k}'\cdot\vec{x}} \right]$$

$$\left(\text{N.B.} \int_{-\infty}^{\infty} dk_x = \int_{\infty}^{-\infty} -dk_x' = \int_{-\infty}^{\infty} dk_x' \quad \Rightarrow \int d^3\vec{k} = \int d^3\vec{k}' \right)$$

Drop the primes

$$\phi^* = \int d^3\vec{k} \left[A^*_{-\vec{k}}\, e^{i\omega t - i\vec{k}\cdot\vec{x}} + B^*_{-\vec{k}}\, e^{-i\omega t + i\vec{k}'\cdot\vec{x}} \right]$$

If ϕ is real, $\phi = \phi^* \Rightarrow B_{\vec{k}} = A^*_{-\vec{k}}, \quad B^*_{-\vec{k}} = A_{\vec{k}}$

$$\phi = \int d^3\vec{k}\left[A_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + A^*_{-\vec{k}}\, e^{i\omega t + i\vec{k}\cdot\vec{x}} \right]$$

$$= \int d^3\vec{k}\left[A_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + A^*_{\vec{k}}\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \right]$$

relabel 2nd term

For later convenience, define

$$a_{\vec{k}} = (2\pi)^3 (2\omega) A_{\vec{k}}$$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left[a_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \right]$$

(as before $\omega = +\sqrt{k^2+m^2}$ is implicit)

Then the time derivative is

$$\dot{\phi}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\omega} \left[-i\omega\, a_{\vec{k}}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} + i\omega\, a^*_{\vec{k}}\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \right]$$

Solution is completely determined by arbitrary coefficients $a_{\vec{k}}$ and $a^*_{\vec{k}}$.

We can solve for $a_{\vec{k}}$ and $a^*_{\vec{k}}$ in terms of ϕ and $\dot{\phi}$ by inverting the equations (using Fourier transform)

(Analogous to harmonic oscillator earlier)

To invert, consider (cf earlier inversion of harmonic osc)

$$I = \int d^3x \, e^{i\omega t - i\vec{k}\cdot\vec{x}} \left[\omega \phi(\vec{x},t) + i\dot{\phi}(\vec{x},t) \right]$$

$$= \int d^3x \, e^{i\omega t - i\vec{k}\cdot\vec{x}} \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[e^{-i\omega' t + i\vec{k}'\cdot\vec{x}} (\omega a_{k'} + \omega' a_{k'}) + e^{i\omega' t - i\vec{k}'\cdot\vec{x}} (\omega a^*_{k'} - \omega' a^*_{k'}) \right]$$

Switch order of integration

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[(\omega + \omega') a_{k'} \, e^{i(\omega-\omega')t} \int d^3x \, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} + (\omega-\omega') a^*_{k'} \, e^{i(\omega+\omega')t} \int d^3x \, e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}} \right]$$

In 3140, we introduce the Dirac delta function δ and prove that

$$\delta^{(3)}(\vec{K}) = \frac{1}{(2\pi)^3} \int d^3x \, e^{i\vec{K}\cdot\vec{x}}$$

So

$$I = \int \frac{d^3k'}{2\omega'} \left[(\omega+\omega') a_{k'} e^{i(\omega-\omega')t} \, \delta(\vec{k}'-\vec{k}) + (\omega-\omega') a^*_{k'} \, e^{i(\omega+\omega')t} \, \delta(-\vec{k}'-\vec{k}) \right]$$

Properties of the Dirac delta function

① vanishes unless argument $= 0$

$$\delta^{(3)}(\vec{k}) = 0 \quad \text{if} \quad \vec{k} \neq 0$$

· therefore when it appears inside an integral it enforces the constraint $\vec{k} = 0$.

② $\to \infty$ if $\vec{k} = 0$ but the integral of $\delta^{(3)}(\vec{k})$ is finite

$$\int d^3\vec{k} \; \delta^{(3)}(\vec{k}) = 1$$

Therefore $\int d^3\vec{k}' \; f(\vec{k}') \, \delta^{(3)}(\vec{k}' - \vec{k})$

$$= \int d^3\vec{k}' \; f(\vec{k}) \, \delta^{(3)}(\vec{k}' - \vec{k})$$

$$= f(\vec{k}) \int \underbrace{d^3\vec{k}'}_{d\vec{K}} \; \delta^{(3)}(\underbrace{\vec{k}' - \vec{k}}_{\vec{K}})$$

$$= f(\vec{k})$$

First term in I $\Rightarrow$ $\vec{k}' = \vec{k}$ $\Rightarrow$ $\omega' = \omega$

$$+ \int \frac{d^3k'}{2\omega'} (\omega + \omega')\, a_k\, e^{i(\omega - \omega')t}\, \delta(\vec{k}' - \vec{k})$$

B : $= a_k \int d^3k'\, \delta(\vec{k}' - \vec{k})$

$= a_k$

Second term in I $\Rightarrow$ $\vec{k}' = -\vec{k}$ $\Rightarrow$ $\omega' = \omega$

term vanishes

$$\Rightarrow a_k = \int d^3x\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \left[\omega \phi(x,t) + i\dot{\phi}(x,t) \right]$$

Take c.c.

$$a_k^* = \int d^3x\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \left[\omega \phi(x,t) - i\dot{\phi}(x,t) \right]$$

[Hw: Show this explicitly]

Clever way to write this

$$a_k = \int d^3x\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \left[i \overleftrightarrow{\partial_0} \phi(x,t) \right]$$

where $f \overleftrightarrow{\partial_0} g = f\, \partial_0 g - (\partial_0 f)\, g$

[cf B.D. footnote to write this covariantly]

Four vectors

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x})$$

The position (up or down) of the indices is important.

A four-vector is defined by how it transforms under a Lorentz transformation

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu$$

Summation convention: Sum over repeated indices (one up + one down)

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$$

Define spacetime interval between two points

$$ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu \qquad \text{where} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Minkowski metric

Particle physicists often use mostly minus metric (West Coast)
Gravitational " " " mostly plus metric (East Coast)

$$\eta_{\mu\nu} = (-1, 1, 1, 1)$$

w/ mostly minus metric

$$ds^2 = dt^2 - (d\vec{x})^2$$

Lorentz transformation $\equiv$ transf. that leaves ds^2 invt.

$$ds^2 = dx'^\mu \eta_{\mu\nu} dx'^\nu = dx^\alpha \eta_{\alpha\beta} dx^\beta$$

$$= (\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta) dx^\alpha dx^\beta$$

$$\Rightarrow \quad \boxed{\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

Sometimes convenient to write as a matrix eqn

$$\underset{\substack{\uparrow \quad \uparrow \\ \text{row} \quad \text{column}}}{(\Lambda)_{\mu\alpha}} = \Lambda^\mu{}_\alpha \qquad \Rightarrow \qquad x' = \underset{\substack{\uparrow \\ 4\times4 \\ \text{matrix}}}{\Lambda}\ \underset{\substack{\uparrow \\ \text{column} \\ \text{vector}}}{x}$$

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$

[Since $\eta^2 = 1 \Rightarrow \quad \Lambda^T \eta \Lambda \eta = 1$

$\Rightarrow \quad \boxed{\eta \Lambda \eta = (\Lambda^T)^{-1}}$] do this later

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Boost in $+x$ direction

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \frac{v}{c} = v \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

A 4-vector is any quantity A^μ that tx's as

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

e.g.

4-velocity $u^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma\vec{\beta})$

4-momentum $p^\mu = mu^\mu = (E, \vec{p})$

4-wavevector $k^\mu = (\omega, \vec{k})$. $\qquad p^\mu = \hbar k^\mu$

$\eta_{\mu\nu} A^\mu B^\nu$ is Lorentz-invariant for any 4-vectors A^μ & B^μ

e.g. $\eta_{\mu\nu} dx^\mu dx^\nu = ds^2$ spacetime interval

$\eta_{\mu\nu} p^\mu p^\nu = E^2 - \vec{p}^{\,2} = m^2$ mass

$\eta_{\mu\nu} k^\mu x^\nu = \omega t - \vec{k}\cdot\vec{x}$ "phase"

$$\phi = e^{i(\vec{k}\cdot\vec{x} - \omega t)} = e^{-i\eta_{\mu\nu} k^\mu x^\nu}$$

A fourvector w/ indices up is called a contravariant 4-vector A^μ

Define the <u>covariant 4-vector</u> $A_\mu \equiv \eta_{\mu\nu} A^\nu$

eg $x_\mu = (t, -\vec{x})$

$k_\mu = (\omega, -\vec{k})$

$p_\mu = (E, -\vec{p})$

[no new information]

Then $\eta_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu = A_\mu B^\mu = A \cdot B$

$\phi = e^{-ik\cdot x}$

$p^2 = m^2$

Define inverse metric $\eta^{\mu\nu}$ as $\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$

$$\delta^{\mu}{}_{\lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

"accidentally" the same as $\eta_{\mu\nu}$

$$A_\mu = \eta_{\mu\nu} A^\nu$$

$$\eta^{\lambda\mu} A_\mu = \eta^{\lambda\mu}\eta_{\mu\nu} A^\nu = \delta^{\lambda}{}_{\nu} A^\nu = A^\lambda$$

so $A^\lambda = \eta^{\lambda\mu} A_\mu$ The metric also raises.

(apologies to Hemingway)

How does a covariant 4-vector transform?

$$A'_\mu = \eta_{\mu\alpha} A^\alpha = \eta_{\mu\alpha} \Lambda^\alpha{}_\beta A^\beta = \eta_{\mu\alpha} \Lambda^\alpha{}_\beta \eta^{\beta\nu} A_\nu$$

$$A'_\mu = \left(\eta_{\mu\alpha} \Lambda^\alpha{}_\beta \eta^{\beta\nu}\right) A_\nu$$

As a matrix eqn:

$A' = \Lambda A$ for contravariant

$A' = \eta \Lambda \eta A$ } for covariant

[$= (\Lambda^T)^{-1} A$] later

<u>Could</u> define: $\Lambda_\mu{}^\nu = \eta_{\mu\alpha} \Lambda^\alpha{}_\beta \eta^{\beta\nu}$

$$\tilde{\Lambda} = \eta \Lambda \eta = (\Lambda^T)^{-1}$$

To see this more directly

$$A'_\mu B'^\mu = \underbrace{\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta}_{\delta^\alpha{}_\beta} A_\alpha B^\beta$$

$$\tilde{\Lambda}^T \Lambda = 1$$

$$\tilde{\Lambda} = (\Lambda^T)^{-1}$$

Thomas:
uses primes
on indices

$$dx^{\mu'} = \Lambda^{\mu'}{}_{\alpha}\, dx^{\alpha}$$

$$\Lambda^{\mu'}{}_{\alpha} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}}$$

<u>Note positions</u>

$$\omega_{\nu'} = \Lambda^{\beta}{}_{\nu'}\, \omega_{\beta} \qquad = \omega_{\beta}\, \Lambda^{\beta}{}_{\nu'}$$

$$\Lambda^{\beta}{}_{\nu'} = \frac{\partial x^{\beta}}{\partial x'^{\nu}}$$

$$\omega_{\mu'}\, dx'^{\mu} = \underbrace{\Lambda^{\beta}{}_{\mu'}\, \Lambda^{\mu'}{}_{\alpha}}_{\delta^{\beta}_{\alpha}}\, \omega_{\beta}\, dx^{\alpha}$$

gradient $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

4-gradient $\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

How does it transform? As a contravariant 4-vector or as a covariant 4-vector? (or neither?)

Act on scalar field

$$\left[\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial \phi}{\partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\right]$$

$$\left(\frac{\partial \phi}{\partial x'^\mu}\right) dx'^\mu = \left(\frac{\partial \phi}{\partial x^\alpha}\right) dx^\alpha \qquad \text{invariant}$$

Since dx^α is contravariant $\frac{\partial \phi}{\partial x^\alpha}$ must be covariant

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~~If that's too slick, try this:~~

$$\cancel{\frac{\partial \phi}{\partial x^\alpha} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial \phi}{\partial x'^\mu} = \Lambda^\mu{}_\alpha \frac{\partial \phi}{\partial x'^\mu}}$$

$$\cancel{\frac{\partial \phi}{\partial x} = \Lambda^T \frac{\partial \phi}{\partial x'}}$$

~~Then~~ $\cancel{\frac{\partial \phi}{\partial x'} = (\Lambda^T)^{-1}\frac{\partial \phi}{\partial x}}$ ~~covariant 4-vector~~

$$\cancel{= (\eta \Lambda \eta)\frac{\partial \phi}{\partial x}}$$

$$\frac{\partial \phi}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial \phi}{\partial x^{\alpha}}$$

$\downarrow$ define

$$= \Omega_{\mu}{}^{\alpha} \frac{\partial \phi}{\partial x^{\alpha}}$$

either do this,
$\boxed{\text{or}}$ assign as a problem
(but w/ specific matrices: boost & rotation)

Now $x'^{\mu} = \Lambda^{\mu}{}_{\alpha} x^{\nu}$

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \Lambda^{\mu}{}_{\alpha}$$

$$\Rightarrow \frac{\partial x^{\alpha}}{\partial x'^{\mu}} = (\Lambda^{-1})^{\alpha}{}_{\mu} = \Omega_{\mu}{}^{\alpha}$$

$$\Rightarrow \Omega = (\Lambda^{-1})^{T}$$

$$\frac{\partial \phi}{\partial x'} = (\Lambda^{-1})^{T} \frac{\partial \phi}{\partial x}$$

But since $\Lambda^{T} \eta \Lambda = \eta$

$$\Lambda^{T} \eta \Lambda \eta = 1$$

$$\Rightarrow (\Lambda^{T})^{-1} = \eta \Lambda \eta$$

$$\Rightarrow \frac{\partial \phi}{\partial x'} = (\eta \Lambda \eta) \frac{\partial \phi}{\partial x}$$ ie transforms a <u>covariant 4-vector</u>

Denote $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu} = \left(\frac{\partial \phi}{\partial t}, \vec{\nabla} \phi \right)$

$$\partial^\mu \phi = \frac{\partial \phi}{\partial x_\mu} = \left(\frac{\partial \phi}{\partial t}, -\vec{\nabla} \phi \right)$$

wave eqn: $0 = \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi$

$$= \underbrace{\underbrace{\eta^{\mu\nu} \partial_\mu \partial_\nu \phi}_{\partial_\mu \partial^\mu \phi}}_{\partial^2 \phi} + m^2 \phi$$

$$(\partial^2 + m^2) \phi = 0$$

FV-9

define $\frac{d^3k}{(2\pi)^3 2\omega} \equiv \widetilde{dk}$

$$\phi(x^\mu) = \int \widetilde{dk} \left[a_k e^{-ik\cdot x} + a_k^* e^{ik\cdot x} \right]\Big|_{k^2=m^2}$$

Prove that $\widetilde{dk}$ is Lz inv't?

$$\int \frac{d^3k}{(2\pi)^3(2\omega)} = \frac{1}{(2\pi)^3} \int d^4k\, \delta(k^2-m^2)\, \theta(k_0)$$

$$\delta((\omega-\omega_k)(\omega+\omega_k))\, \theta(\omega)$$

$$= \frac{1}{2\omega_k} \delta(\omega-\omega_k)$$

$\boxed{\text{Lagrangian approach to fields}}$

First, review harmonic oscillator

$$S = \int dt\, L = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \right) dt \qquad \text{(set } m = 1\text{)}$$

Functional derivative

$$\frac{\delta S}{\delta x} = \int_{t_0}^{t_1} dt \left(\dot{x}\,\delta\dot{x} - \omega^2 x\, \delta x \right) \qquad \delta\dot{x} = \delta \frac{dx}{dt} = \frac{d}{dt}\delta x$$

IBP

$$= \int_{t_0}^{t_1} dt \left(-\ddot{x}\,\delta x - \omega^2 x\, \delta x \right) + \underbrace{\dot{x}\,\delta x \Big|_{t_0}^{t_1}}_{0}$$

$$= \int dt \left(\ddot{x} + \omega^2 x \right) \delta x \qquad \text{because } \begin{cases} \delta x(t_0) = 0 \\ \delta x(t_1) = 0 \end{cases}$$

$$\frac{\delta S}{\delta x} = 0 \text{ for any variation } \delta x \implies \ddot{x} + \omega^2 x = 0$$

For a field $\phi(x,t)$

$$S = \int dt\, L = \int dt \int d^3x\, \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x\, \mathcal{L}$$

Lagrangian density

d^4x invt
$d^4x' = |\det \Lambda|\, d^4x$
so $\mathcal{L}$ invt

Kinetic contribution: $\frac{1}{2}\dot{\phi}^2$

$\mathcal{L}$ must be a Lorentz-invariant scalar

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2$$

Add a potential energy term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

e.g. $V(\phi) = \frac{1}{2}m^2\phi^2$ for a free massive scalar field

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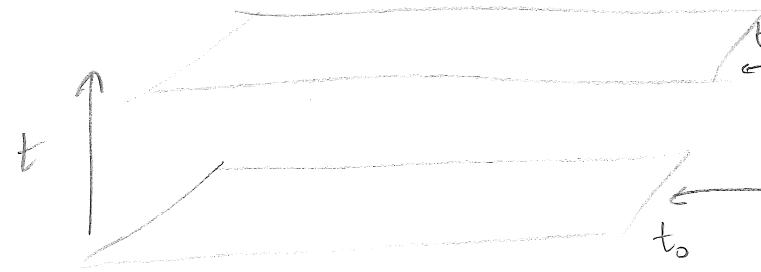
Define canonical momentum density

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

More generally define $\pi^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$

Principle of least action:

action is an extremum for all fields that have fixed profiles at t_0 & t_1 (and vanish at spatial infinity)



$$\phi(t,\vec{x}) \xrightarrow[|x|\to\infty]{} 0$$

The functional variation of S under $\delta\phi$ w/ $\begin{cases} \delta\phi(t_0,x) = 0 \\ \delta\phi(t_1,x) = 0 \\ \delta\phi(x) \xrightarrow[|\vec{x}|\to\infty]{} 0 \end{cases}$

$S[\phi, \partial_\mu\phi]$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right]$$

(justification below)

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \pi^\mu \partial_\mu \delta\phi \right]$$

$\downarrow$ IBP

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\pi^\mu\,\delta\phi + \partial_\mu(\pi^\mu\delta\phi)\right]$$

[Aside: $\delta(\partial_\mu\phi) \equiv \frac{\partial(\phi+\delta\phi)}{\partial x^\mu} - \frac{\partial\phi}{\partial x^\mu} = \partial_\mu(\delta\phi)$]

cf B+D p. 14

Consider

$$\int d^4x\, \partial_\mu (\pi^\mu \delta\phi) = \int d^3x \underbrace{\int_{t_0}^{t_1} dt \frac{d}{dt}(\pi^0 \delta\phi)}_{\pi^0 \delta\phi \Big|_{t_0}^{t_1} \to 0} + \int dt \underbrace{\int d^3x\, \vec{\nabla} \cdot (\vec{\pi}\, \delta\phi)}_{\oint_{\text{sphere at } \infty} \vec{\pi}\, \delta\phi \cdot d\vec{A} \to 0}$$

Boundary term vanishes

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \pi^\mu \right] \delta\phi = 0$$

for any $\delta\phi$

$\Rightarrow$ Euler-Lagrange eqns for the field

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\cancel{\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi = 0}$$

~~for any variation~~ $\cancel{\delta\phi}$

$\Rightarrow$ ~~Euler-Lagrange eqn~~

$$\cancel{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$

For $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu \phi)} \left[\frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi)\, \delta^\mu_\beta + \frac{1}{2} \eta^{\alpha\beta} \delta^\mu_\alpha (\partial_\beta \phi)$$

$$= \frac{1}{2} \partial^\mu \phi + \frac{1}{2} \partial^\mu \phi = \partial^\mu \phi$$

$$\Rightarrow \quad -\frac{\partial V}{\partial \phi} - \partial_\mu (\partial^\mu \phi) = 0$$

ie $\quad \partial^2 \phi + \dfrac{\partial V}{\partial \phi} = 0$

$$\ddot{\phi} - \nabla^2 \phi + \frac{\partial V}{\partial \phi} = 0$$

Free massive $V = \frac{1}{2} m^2 \phi^2$

$$(\partial^2 + m^2)\phi = 0$$

Energy of a scalar field

Recall for a particle L = (kinetic) − (potential) = $\frac{1}{2}\dot{x}^2 - V(x)$

E = (kinetic) + (potential) = $\frac{1}{2}\dot{x}^2 + V(x)$

Therefore can write

$$E = \dot{x}^2 - L$$

For a scalar field $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$

Lagrange density

$$= \underbrace{\frac{1}{2}\dot{\phi}^2}_{\text{kinetic}} - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

Let us guess (proof later) that the energy density is

$$\rho_E = \dot{\phi}^2 - \mathcal{L}$$

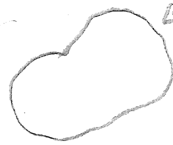
$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

Recall from P3000 that the density of <u>conserved</u> quantity is related to the current of that quantity by a "continuity equation"

Let ρ = density (per volume) e.g. of charge

Let $\vec{J}$ = current density (per area)

arbitrary volume



Let Q = charge in the volume $= \int_V d^3x\, \rho$

Flow of quantity out of the volume $= \oint_{\partial V} \vec{J} \cdot d\vec{A}$

Change of quantity inside the volume

$$\dot{Q} = \int d^3x \frac{\partial \rho}{\partial t} = -\oint \vec{J} \cdot d\vec{A}$$

because quantity is conserved

But $\oint_{\partial V} \vec{J} \cdot d\vec{A} \equiv \int d^3x\, \vec{\nabla} \cdot \vec{J}$ [Gauss]

$$\Rightarrow \int d^3x \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right] = 0$$

Since volume is arbitrary $\Rightarrow$ $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ (continuity eqn)

Also, if volume is infinite $\dot{Q} = 0 \Rightarrow Q = \text{const}$

Rewrite continuity eqn in Lorentz covariant form

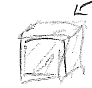
$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = \partial_\mu J^\mu \qquad \text{where } J^\mu = (\rho, \vec{J})$$

This is valid provided J^μ transforms as a 4-vector.
w/o a full proof, let's see that this is *plausible*

$$J'^\mu = \Lambda^\mu{}_\nu J^\nu$$

Consider a boost in $+x$ direction $\Lambda = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$\rho' = \gamma\rho - \gamma v J_x$$
$$J'_x = -\gamma v \rho + \gamma J_x$$



charge in box $= \rho\, dx\, dy\, dz$

Suppose in unprimed frame $J_x = 0$

then in primed frame $\rho' = \gamma\rho$, $J'_x = -\gamma v \rho$ ✓

box is Lorentz contracted by factor γ

$\rho \uparrow$ by γ

$$Q = \int \rho\, d^3x$$

$\uparrow$

Lorentz invariant

For a scalar field

$$\rho_E = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

Assuming that energy is a conserved quantity (prove later), what is the associated energy current density?

$$\frac{\partial \rho_E}{\partial t} = \dot{\phi}\ddot{\phi} + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\dot{\phi}) + \frac{\partial V}{\partial \phi}\dot{\phi}$$

E-L eqn: $$\ddot{\phi} = \nabla^2\phi - \frac{\partial V}{\partial \phi}$$

$$\frac{\partial \rho_E}{\partial t} = \dot{\phi}\nabla^2\phi + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\dot{\phi})$$

$$= \vec{\nabla}\cdot(\dot{\phi}\vec{\nabla}\phi)$$

$$\Rightarrow \vec{J}_E = -\dot{\phi}\vec{\nabla}\phi$$

Can we then write

$$j_E^\mu = (j_E^0, \vec{j}_E)$$

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Then $E = \int d^3x \, j_E^0$ is Lorentz invariant

But E is part of a 4-vector!

$$p^\nu = (E, \vec{p})$$

This suggests

$$p^\nu = \int d^3x \, T^{0\nu}$$

$$p^0 = \int d^3x \, T^{00},$$

$$p^j = \int d^3x \, T^{0j},$$

$T^{\mu\nu}$ = energy-momentum <u>tensor</u>

T^{00} = energy density ρ_E

T^{0j} = momentum density

T^{i0} = energy current j_E^i

T^{ij} = momentum current

$T^{\mu\nu}$ = 2nd rank tensor, transforms as

$$T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$$

For scalar field

$$T^{00} = \dot{\phi}^2 - \mathcal{L} = (\partial^0\phi)(\partial^0\phi) - \eta^{00}\mathcal{L}$$

$$T^{i0} = -\dot{\phi}\,\partial_i\phi = (\partial^i\phi)(\partial^0\phi)$$

This suggests

$$\underline{T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}}$$

Energy momentum conservation implies continuity eqn

$\partial_\mu T^{\mu\nu} = 0$ [can verify using E-L eqns]

prob

$\cancel{\partial^2\phi\,\partial^\nu\phi - \eta^{\mu\nu}[\partial_\lambda\phi\,\partial}$

$$\partial_\mu T^{\mu\nu} = \partial^2\phi\,\partial^\nu\phi + \partial^\mu\phi\,\partial_\mu\partial^\nu\phi \underbrace{- \partial^\nu\left[\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2\right]}_{-\partial^\nu\partial^\lambda\phi\,\partial_\lambda\phi \,+\, m^2\phi\,\partial^\nu\phi}$$

$\partial^2\phi$

Noether's thm: symmetries of the Lagrangian implies the existence of conserved currents

Action is invariant under spacetime translations

$$x^\nu \longrightarrow x^\nu + a^\nu$$
$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x+a) = \mathcal{L}(x) + a_\nu \partial^\nu \mathcal{L} + O(a^2)$$

$\mathcal{L}$ only depends on spacetime crds through its dependence on the fields

$$\mathcal{L}(\phi(x)) \longrightarrow \mathcal{L}(\phi(x+a))$$
$$= \mathcal{L}(\phi(x)) + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \text{ by euler-lagrange}} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu \delta\phi}$$

$$= \mathcal{L}(\phi(x)) + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]$$

$$\delta\phi = \phi(x+a) - \phi(x) = a_\nu \partial^\nu \phi + O(a^2)$$

$$= \mathcal{L}(\phi(x)) + a_\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \right]$$

But we saw above $\mathcal{L} \longrightarrow \mathcal{L}(x) + a_\nu \partial_\mu (\eta^{\mu\nu} \mathcal{L})$

so

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \right] = 0$$

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then the canonical ~~energy-mom tensor~~ is

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} \quad \Rightarrow \quad \partial_\mu T^{\mu\nu} = 0}$$

For $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$ this is just

$$= \partial^\mu \phi\, \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}$$

just as we surmised.

$$\partial_\mu T^{\mu\nu} = 0$$

$$p^\nu = \int d^3x\, T^{0\nu}$$

$$\frac{dp^\nu}{dt} = \int d^3x\, \partial_0 T^{0\nu} = \int d^3x\,(-\partial_i T^{i\nu})$$

$$= -\oint_{\substack{\text{sphere}\\ \text{at } \infty}} dA^i\, T^{i\nu} = 0$$