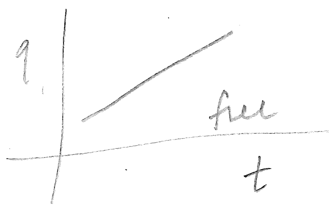
$\boxed{\text{Lagrangian approach to mechanics}}$

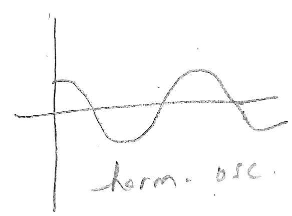
$\boxed{\text{Principle of least action}}$

ref: any advanced mechanics textbook

[Read: Feynman lectures vol II, ch 19-1 thru 19-8.]

trajectory of a point particle $q(t)$ [q = generalized coordinate]





Newtonian approach: $q(t)$ obeys 2nd order o.d.e. $m\ddot{q} = F$
with initial conditions $q(t_0)$, $\dot{q}(t_0)$

Lagrangian approach: $q(t)$ extremizes the action functional S
subject to fixed endpoints
$q(t_0)$ and $q(t_1)$ "principle of least action"

[function: # $\to$ #
functional: $f(x) \to$ #] $\longrightarrow$ "functional" = function of a function
$S[q(t)]$

Functional derivative $\dfrac{\delta S}{\delta q} = 0$, extremum

"calculus of variations"

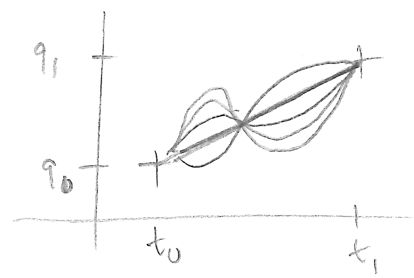
Q: What does $\dfrac{\delta}{\delta q}$ mean??

Let $q(t)$ be the trajectory that minimizes S w/ $q(t_0) = q_0$ + $q(t_1) = q_1$

Let $\eta(t)$ be an arbitrary function w/ $\eta(t_0) = 0$ and $\eta(t_1) = 0$.

Define $q_\epsilon(t) = q(t) + \epsilon\eta(t)$, a family of trajectories with the same endpoints $q_\epsilon(t_0) = q_0$ and $q_\epsilon(t_1) = q_1$:

$$S[q_\epsilon(t)] \geq S[q(t)] \quad \text{for all } \epsilon \neq 0 \text{ and all } \eta.$$



$$\underbrace{\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = 0}_{\text{principle of least (extremal) action}}$$

S is the integral of a function of q and $\dot{q}$ called the $\boxed{\text{Lagrangian}}$

(Joseph-Louis Lagrange) 1736-1813 Italian

$$S[q(t)] = \int_{t_0}^{t_1} dt \; L(q, \dot{q})$$

$$S[q_\epsilon(t)] = \int_{t_0}^{t_1} dt \; L(q + \epsilon\eta, \dot{q} + \epsilon\dot{\eta})$$

$$\frac{dS}{d\epsilon} = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \right)$$

[Recall IBP $\quad \frac{d}{dt}(ab) = a\dot{b} + \dot{a}b$

$$\Rightarrow \int dt \; a\dot{b} = -\int dt \; \dot{a}b + \int dt \frac{d}{dt}(ab)$$]

$$\frac{dS}{dt} = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q}\eta - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\eta \right) + \underbrace{\int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\eta\right)}_{\underbrace{\frac{\partial L}{\partial \dot{q}}\eta \Big|_{t_0}^{t_1}}_{0}}$$

$$= \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right)\eta$$

The only way this can vanish for any function $\eta(t)$ is

if $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad$ for all $t_0 < t < t_1$

Euler-Lagrange equation

[normally, $\frac{dy}{dx} = 0 \Rightarrow$ pt

but calc of var'n $\Rightarrow$ eqn]

For a non-relativistic system:

$$L = (\text{kinetic energy}) - (\text{potential energy})$$

one-dim'l point particle

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

Define $\boxed{\text{canonical momentum}}$ $p \equiv \frac{\partial L}{\partial \dot{q}}$. For pt-pcl, $p = m\dot{q}$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m\ddot{q} + \underbrace{\frac{\partial V}{\partial q}}_{-(\text{Force})}$$

$\boxed{\text{Noether's theorem: continuous symmetry} \Rightarrow \text{conservation laws}}$

e.g. invariance of laws of physics under translation in time $t \to t + a$ implies conservation of energy.

$L(q(t), \dot{q}(t))$ Lagrangian only depends on time through its dependence on configuration variables; it does not depend explicitly on time

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial q}} \frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}} \frac{d\dot{q}}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \dot{q} + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\dot{q})$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q}\right)$$

$$0 = \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{q}} \dot{q} - L}_{\text{conserved quantity } (= E)}\Big) = 0$$

If $L = \frac{1}{2} m \dot{q}^2 - V(q)$ then $E = \frac{1}{2} m \dot{q}^2 + V(q)$

$\boxed{\text{Hamiltonian}}$ H = energy written in terms of $p = m\dot{q} \Rightarrow H = \frac{p^2}{2m} + V(q)$

Spatial Translation invariance $\Rightarrow$ cons of moment

2 interacting particles $V(x_1 - x_2)$

x_1, x_2

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - V(x_1 - x_2)$$

Observe that L is invt under overall translation in space

$$x_1' = x_1 + a \qquad \dot{x}_1' = \dot{x}_1$$

$$x_2' = x_2 + a \qquad \dot{x}_2' = \dot{x}_2$$

$$x_1' - x_2' = x_1 - x_2$$

Let $L_a = L(x_1', \dot{x}_1', x_2', \dot{x}_2') = L(x_1, \dot{x}_1, x_2, \dot{x}_2) = L_0$

ie $\dfrac{dL_a}{da} = 0$

true for any # of particles interacting via ~~2-body~~ relative positions

chain rule

$$\left.\frac{dL_a}{da}\right|_{a=0} = \frac{\partial L}{\partial x_1'} \underbrace{\frac{\partial x_1'}{\partial a}}_{1} + \frac{\partial L}{\partial \dot{x}_1'} \underbrace{\frac{\partial \dot{x}_1'}{\partial a}}_{0} + \frac{\partial L}{\partial x_2'} \underbrace{\frac{\partial x_2'}{\partial a}}_{1} + \frac{\partial L}{\partial \dot{x}_2'} \underbrace{\frac{\partial \dot{x}_2'}{\partial a}}_{0}$$

$$= \frac{\partial L}{\partial x_1} + \frac{\partial L}{\partial x_2}$$

use E-L

$$= \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{x}_1}}_{p_1}\Big) + \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{x}_2}}_{p_2}\Big) = \frac{d}{dt}(p_1 + p_2) = 0$$

ie total mom $p_1 + p_2$ is conserved

Classical harmonic oscillator

Harmonic oscillator

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \qquad k = m\omega^2$$

w/o loss of generality, set $m = 1$. Just setting the scale

[How w/o loss? What does $m=1$ mean since dim'l param?

$m = 1$ kg $\Rightarrow$ setting kg to be unit.]

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$$

E-L $\Rightarrow$ $\ddot{x} + \omega^2 x = 0$

2nd order o.de $\Rightarrow$ 2 indep solutions $e^{\pm i\omega t}$

most general solution $\quad x(t) = x_+ e^{-i\omega t} + x_- e^{i\omega t}$

complex coeffs (initial conditions)

$x(t)$ real $\Rightarrow$ $x_- = x_+^*$

$$\dot{q} = -i\omega q_+ e^{-i\omega t} + i\omega q_- e^{i\omega t}$$

$$\left.\begin{matrix} q(t) \\ \dot{q}(t) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} q_+ \\ q_- \end{matrix}\right.$$

Solve for $q_\pm$:

$$\begin{cases} q_+ = \frac{1}{2} e^{i\omega t}\left[q(t) + \frac{i}{\omega}\dot{q}(t)\right] \\ q_- = \frac{1}{2} e^{-i\omega t}\left[q(t) - \frac{i}{\omega}\dot{q}(t)\right] \end{cases}$$

Define $a = \sqrt{2\omega}\, x +$

$$\begin{cases} x(t) = \frac{1}{\sqrt{2\omega}}\, a\, e^{-i\omega t} + \frac{1}{\sqrt{2\omega}}\, a^* e^{i\omega t} \\ \dot{x}(t) = -i\sqrt{\frac{\omega}{2}}\, a\, e^{-i\omega t} + i\sqrt{\frac{\omega}{2}}\, a^* e^{i\omega t} \end{cases}$$

Solve for a, a^* in terms of $q(t), \dot{q}(t)$

$$\begin{cases} a = e^{i\omega t}\left[\sqrt{\frac{\omega}{2}}\, x(t) + \frac{i}{\sqrt{2\omega}}\, \dot{x}(t)\right] \\ a^* = e^{-i\omega t}\left[\sqrt{\frac{\omega}{2}}\, x(t) - \frac{i}{\sqrt{2\omega}}\, \dot{x}(t)\right] \end{cases}$$

<u>Side remark:</u>

$$a = \frac{i}{\sqrt{2\omega}}\left[e^{i\omega t}\, \overleftrightarrow{\frac{d}{dt}}\, x(t)\right] \qquad \overleftrightarrow{\frac{d}{dt}} = \overrightarrow{\frac{d}{dt}} - \overleftarrow{\frac{d}{dt}}$$

$$a^* = -\frac{i}{\sqrt{2\omega}}\left[e^{-i\omega t}\, \overleftrightarrow{\frac{d}{dt}}\, x(t)\right]$$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

$$= \frac{1}{2}\left(\frac{\omega}{2}\right)\left(-a^2 e^{-2i\omega t} + aa^* + a^*a - a^{*2} e^{2i\omega t}\right)$$

$$+ \frac{1}{2}\omega^2\left(\frac{1}{2\omega}\right)\left(a^2 e^{-2i\omega t} + aa^* + a^*a + a^{*2} e^{2i\omega t}\right)$$

$$= \frac{\omega}{2}\left(aa^* + a^*a\right)$$

indeed <u>indep</u> of time!

we'll use this later

Quantum mechanics: Schrödinger picture

state of system $\psi(x,t)$ — more abstractly: ket $|\psi(t)\rangle$

complex conjg $\psi^*(x,t)$ — abstractly bra $\langle\psi(t)|$

Probability $\int dx\, \psi^*(x,t)\, \psi(x,t) = 1$ — bracket $\langle\psi(t)|\psi(t)\rangle = 1$

$\uparrow$

bracket implies integration over x.

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operators: position x — X

momentum $\frac{\hbar}{i}\frac{d}{dx}$ — P

Hamiltonian $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$ — $H = \frac{P^2}{2m} + V(x)$

expectation values of an operator

$$\langle A(t)\rangle = \int dx\, \psi^*(x,t)\, \hat{A}\, \psi(x,t) := \langle\psi(t)|A|\psi(t)\rangle$$

operators do not commute in general

$$xp \neq px$$

Define $\boxed{\text{commutator}}$ $[A, B] = AB - BA$

measure of noncommutativity of operators

eg. $[x, p]\psi = x\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi - \frac{\hbar}{i}\frac{d}{dx}(x\psi) = -\frac{\hbar}{i}\psi$

$$\boxed{[X, P] = i\hbar\mathbb{1}}$$

$\mathbb{1}$ = identity operator

$\mathbb{1}|\psi\rangle = |\psi\rangle$

Important commutator identity

$$[O, AB\ldots Z] = [O, A]B\ldots Z + A[O, B]\ldots Z + \ldots + AB\ldots[O, Z]$$

[Prove]

$\boxed{\text{+ internalize}}$

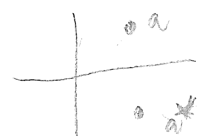
Compute $[AB, CD]$

Incidentally $[A, A] = 0$

$\boxed{\text{Hermitian conjugate of an operator: } A^{\dagger}}$

$A \to A^{\dagger}$ $\qquad\qquad$ $a \to a^{*}$

h.c. is to operators as c.c is to numbers



If $\quad |\phi\rangle = A\,|\psi\rangle$

then $\quad \langle\phi| = \langle\psi|\,A^{\dagger}$

real numbers corresp to hermitian operators

$a^{*} = a$ $\qquad\qquad$ $A^{\dagger} = A$

imag numbers corresp to anti hermitian operators

$A^{\dagger} = -A$

$a^{*} = -a$

All observables (position, momentum, energy) are real
Corresponding operators are hermitian

$\Rightarrow \; X^{\dagger} = X$

$p^{\dagger} = p$

$H^{\dagger} = H$

Time evolution

Schrodinger eqn dictates time evolution of the state

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

solution $|\psi(t)\rangle = U(t) |\psi(0)\rangle$

where $U(t) = e^{-\frac{iHt}{\hbar}} \equiv \sum_0^\infty \frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^n$

proof: $$\frac{dU}{dt} = \sum_{n=1}^\infty \frac{n}{n!}\left(-\frac{iHt}{\hbar}\right)^{n-1}\left(-\frac{iH}{\hbar}\right)$$

$$= -\frac{iH}{\hbar}\sum_{n'=0}^\infty \frac{1}{n'!}\left(-\frac{iHt}{\hbar}\right)^{n'} \qquad n' = n-1$$

$$= -\frac{iH}{\hbar} U(t)$$

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then $$i\hbar \frac{d}{dt} \underbrace{U(t)|\psi(0)\rangle}_{|\psi(t)\rangle} = H \underbrace{U(t)|\psi(0)\rangle}_{|\psi(t)\rangle}$$

$\langle\psi(t)| = \langle\psi(0)| U^\dagger$

$$U^\dagger = \sum \frac{1}{n!}\left(\frac{(-i)^* H^\dagger t}{\hbar}\right)^n = \sum \frac{1}{n!}\left(\frac{iHt}{\hbar}\right)^n = e^{\frac{iHt}{\hbar}}$$

Now $U^\dagger U = e^{\frac{iHt}{\hbar}} e^{-\frac{iHt}{\hbar}} = e^0 = \mathbb{1} \Rightarrow U$ is a unitary operator

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U^\dagger U|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

A normalized state remains normalized

$U(t)$ is the unitary operator that implements time translation

Schrodinger vs Heisenberg pictures

HP #1

In the Schrodinger picture, operators are independent of time; the state carries the time dependence.

In the Heisenberg picture, the situation is reversed. The <u>state</u> is given by its value at $t=0$ in the Schrodinger picture

$$|\psi\rangle_H = |\psi(0)\rangle_S$$

Consider the observable $\langle A(t)\rangle$, which is the same in both pictures

$$\langle A(t)\rangle = {}_S\langle\psi(t)|\, A_S\, |\psi(t)\rangle_S$$

$$= {}_S\langle\psi(0)|\, U^\dagger(t)\, A_S\, U(t)\, |\psi(0)\rangle_S$$

Define the operator in the Heisenberg picture

$$A_H(t) = U^\dagger(t)\, A_S\, U(t)$$

$$\langle A(t)\rangle = {}_H\langle\psi|\, A_H(t)\, |\psi\rangle_H$$

(closer to classical mechanics)

[ref: Sakurai, <u>Modern QM</u>]

$$A_H(t+a) = e^{\frac{iH(t+a)}{\hbar}} A \, e^{-\frac{iH(t+a)}{\hbar}}$$

$$= e^{\frac{iHa}{\hbar}} e^{\frac{iHt}{\hbar}} A \, e^{-\frac{iHt}{\hbar}} e^{-\frac{iHa}{\hbar}}$$

$$= e^{\frac{iHa}{\hbar}} A(t) \, e^{-\frac{iHa}{\hbar}}$$

$$A_H(t+a) = U^\dagger(a) A_H(t) U(a)$$

classical: time translation invariance $\Rightarrow$ H is conserved

quantum: operator H "generates" time translation

via the unitary operator $U(t) = e^{-\frac{iHt}{\hbar}}$

N.B. operator H is indep of time

$$H_H(t) = e^{\frac{iHt}{\hbar}} H \, e^{-\frac{iHt}{\hbar}} = H$$

From now on, all ops will be in Heisenberg picture

$$\frac{d}{dt}A(t) = \frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} A(0)\, e^{-\frac{iHt}{\hbar}}\right)$$

$$= \frac{i}{\hbar} H \underbrace{e^{\frac{iHt}{\hbar}} A(0)\, e^{-\frac{iHt}{\hbar}}}_{A(t)} - \frac{i}{\hbar} \underbrace{e^{\frac{iHt}{\hbar}} A(0)\, e^{-\frac{iHt}{\hbar}}}_{A(t)} H$$

$$= -\frac{i}{\hbar}[A(t), H]$$

$$\boxed{i\hbar \frac{d}{dt}A(t) = [A(t), H]}$$

Heisenberg eqn (for Heisenberg picture ops)

[written first by Dirac [Sakurai; p89]]

Conserved quantity A $\Leftrightarrow$ operator $\hat{A}$ commutes w/ H

$[H, H] = 0 \Rightarrow H$ conserved

Consider commutator in Heisenberg picture

$[x_H(t), p_H(t)]$ ← "equal time commutator"

$= [U^+ x_s U, U^+ p_s U]$

$= U^+ \underbrace{[x_s, p_s]}_{i\hbar} U$

$= i\hbar \mathbb{1}$

But $\underbrace{[x_H(t), p_H(t')]}_{\text{unequal time}} \neq i\hbar$ for $t' \neq t$

Also $[x_H(t), x_H(t)] = 0$

but $[x_H(t), x_H(t')] \neq 0$ for $t' \neq t$

[You will calculate these]

Recall for a particle in a potential

$$H = \frac{p^2}{2m} + V(x) = \frac{p(0)^2}{2m} + V(x(0))$$

Schrodinger picture Heisenberg picture

Since H is indep of time

$$H = \frac{p(t)^2}{2m} + V(x(t))$$

Consider

$$[x(t), H] = \frac{1}{2m}\underbrace{[x(t), p^2(t)]}_{[x(t),p(t)]p(t) + p(t)[x(t),p(t)]} + \underbrace{[x(t), V(x(t))]}_{0}$$

$$[\underbrace{x(t), p(t)}_{i\hbar}]\,p(t) + p(t)\,[\underbrace{x(t)\,p(t)}_{i\hbar}]$$

$$= \frac{i\hbar}{m}\,p(t)$$

But $i\hbar \frac{dx(t)}{dt} = [x(t), H]$ so

$$p(t) = m\,\frac{dx(t)}{dt}$$

operator eqn same as classical relation

~~Skip~~ $\longleftrightarrow$ they work it out in hmwk problem

$$[p(t), H] = \left[p(t), \frac{p(t)^2}{2m} + V(x(t))\right]$$

$$= [p(t), V(x(t))]$$

$$= \frac{\hbar}{i} \frac{dV}{dx}(x(t))$$

$$\frac{dp(t)}{dt} = -\frac{dV}{dx}(x(t))$$

$$\cancel{[p(t), H] = i\hbar \frac{dp(t)}{dt}}$$

$$m \frac{d^2 x(t)}{dt^2} = -\frac{\partial V}{\partial x}(x(t))$$

$$\cancel{[A(t), H] = i\hbar \frac{\partial A(t)}{\partial t}}$$

Heisenberg operators obey classical eqns of motion

$\boxed{\text{Quantum oscillator}}$

$$H = \frac{1}{2}p(t)^2 + \frac{1}{2}\omega^2 x(t)^2 \qquad (m=1 \text{ as before})$$

From now on, set $\hbar = 1$

$$[x(t), p(t)] = i$$

Consider

$$[p(t), H] = \frac{1}{2}\omega^2 [p(t), x(t)^2]$$
$$= \frac{1}{2}\omega^2 \, 2x(t)(-i)$$
$$= -i\omega^2 x(t)$$

$$\frac{dp}{dt} = -i[p(t), H] = -\omega^2 x(t)$$

$$\Rightarrow \ddot{x}(t) = -\omega^2 x(t) \qquad \text{same as classical e.o.m.}$$

Define

$$\left.\begin{aligned} a(t) &= \sqrt{\frac{\omega}{2}}\, x(t) + \frac{i}{\sqrt{2\omega}}\, p(t) \\ a^\dagger(t) &= \sqrt{\frac{\omega}{2}}\, x(t) - \frac{i}{\sqrt{2\omega}}\, p(t) \end{aligned}\right\} \qquad \begin{aligned} x(t) &= \frac{1}{\sqrt{2\omega}}\left[a(t) + a^\dagger(t)\right] \\ p(t) &= -i\sqrt{\frac{\omega}{2}}\left[a(t) - a^\dagger(t)\right] \end{aligned}$$

Then $[a(t), a^\dagger(t)] = \sqrt{\frac{\omega}{2}}\left(-\frac{i}{\sqrt{2\omega}}\right) i + \frac{i}{\sqrt{2\omega}}\sqrt{\frac{\omega}{2}}\,(-i) = 1$

Since $[x(t), H] = i p(t)$

and $[p(t), H] = -i\omega^2 x(t)$

$$\Rightarrow [a(t), H] = \sqrt{\frac{\omega}{2}}\, i p(t) + \frac{i}{\sqrt{2\omega}}(-i\omega^2)\, x(t) = \omega a(t)$$

$$[a^\dagger(t), H] = -\omega a^\dagger(t)$$

Since $\frac{da}{dt} = -i[a, H] = -i\omega a(t)$

$$\Rightarrow a(t) = \underbrace{a(0)}_{a}\, e^{-i\omega t}$$

Also $a^\dagger(t) = \underbrace{a^\dagger(0)}_{a^\dagger}\, e^{i\omega t}$

$$\left.\begin{aligned} a &= e^{i\omega t}\left[\sqrt{\frac{\omega}{2}}\, x(t) + \frac{i}{\sqrt{2\omega}}\, p(t)\right] \\ a^\dagger &= e^{-i\omega t}\left[\sqrt{\frac{\omega}{2}}\, x(t) - \frac{i}{\sqrt{2\omega}}\, p(t)\right] \end{aligned}\right. \Rightarrow \begin{aligned} x(t) &= \frac{1}{\sqrt{2\omega}}\left[a e^{-i\omega t} + a^\dagger e^{i\omega t}\right] \\ p(t) &= -i\sqrt{\frac{\omega}{2}}\left[a e^{-i\omega t} - a^\dagger e^{i\omega t}\right] \end{aligned}$$

$$H = \frac{\omega}{2}[a^\dagger a + a a^\dagger] = \frac{\omega}{2}[a^\dagger a + \underbrace{[a, a^\dagger]}_{1} + a^\dagger a] = (a^\dagger a + \tfrac{1}{2})\omega$$

$$[H, a^\dagger] = \omega a^\dagger \quad \Rightarrow \quad H a^\dagger = a^\dagger (H + \omega)$$

$$[H, a] = -\omega a \quad \Rightarrow \quad H a = a (H - \omega)$$

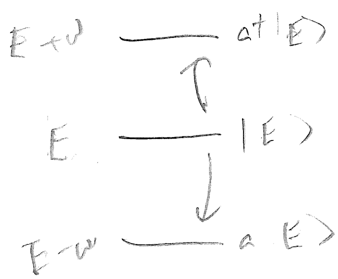
Suppose $|E\rangle$ is an eigenstate of H

$$H|E\rangle = E_1 |E\rangle$$

then $a^\dagger |E\rangle$ and $a|E\rangle$ are also eigenstates

Proof:
$$H(a^\dagger |E\rangle) = a^\dagger (H + \omega)|E\rangle = a^\dagger (E + \omega)|E\rangle$$
$$= (E_1 + \omega)(a^\dagger |E\rangle)$$

$$H(a|E\rangle) = (E_1 - \omega)(a|E\rangle)$$



ladder ops.

not the energy

$\exists$ a lowest state, call it $|0\rangle$

(otherwise a would lower energy indefinitely)

$$a|0\rangle = 0$$

$$H|0\rangle = \left(\omega a^\dagger a + \frac{\omega}{2}\right)|0\rangle = \frac{\omega}{2}|0\rangle \quad \Rightarrow \quad E_0 = \frac{\omega}{2}$$

Define $|n\rangle \sim (a^\dagger)^n |0\rangle$ so $E_n = (n + \frac{1}{2})\omega$

In particular $|n\rangle = K_n a^\dagger |n-1\rangle$

where K_n is chosen so that all states are normalized [HW prob]

$[K_n = \frac{1}{\sqrt{n}}]$

Then $$|n\rangle = \frac{(a^\dagger)^n}{\prod_{m=1}^{n} K_m} |0\rangle$$