# Quantum mechanics

$\boxed{\text{State of a system}}$ (eg particle, atom, molecule, ping pong ball)

is described by a ket $|\psi\rangle \in$ complex vector space (Hilbert space)

bracket $\underbrace{\langle\phi|}_{\text{bra}}\underbrace{\psi\rangle}_{\text{ket}}$

vector space $\Rightarrow$ if $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states
so are all complex linear combinations

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle \qquad c_i \in \mathbb{C}$$

[differs from classical physics]

Can write either $c|\psi\rangle$ or $|\psi\rangle c$

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Postulates:

- all info about the state is contained in $|\psi\rangle$
  (no additional "hidden variables")
- $|\psi\rangle$ and $c|\psi\rangle$ (for any $c \in \mathbb{C}$) describe the <u>same</u> physical state

## observable A

some measurable quantity of the system

eg $S_z$ : z-component of spin

$X$ : x-component of position

$E$ : energy of a particle or of a system

[use capital letters]

## Spectrum of the observable A

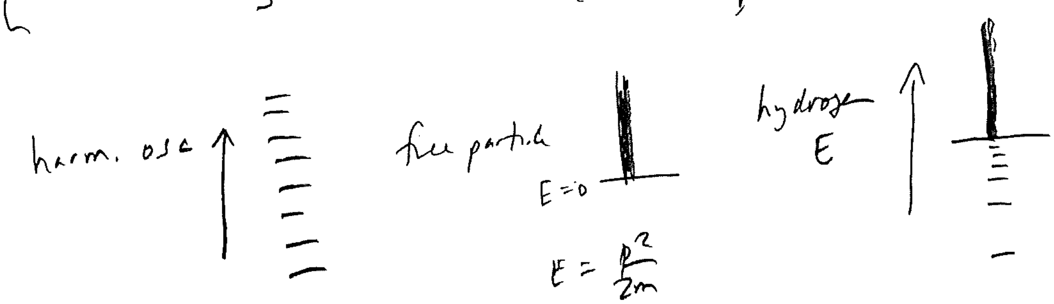
set of possible results of a measurement of A (always real numbers), $\{a\}$

[different meaning than spectra emitted by an atom]

[some observables have] discrete spectra ($S_z$: $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$)

[some observables have] continuous spectra ($X$: $-\infty$ to $\infty$)

[some have] both ($E$: bound states discrete, free states continuous)




$E = \hbar\omega(n+\frac{1}{2})$

## determinate states of an observable A

Certain states $|a\rangle$, upon measurement of A, yield a definite (determinate) result, $a$

e.g. $S_z$ has two determinate states ("eigenstates")

$|S_z = \frac{\hbar}{2}\rangle = |+\rangle$ "spin up"

$|S_z = -\frac{\hbar}{2}\rangle = |-\rangle$ "spin down"

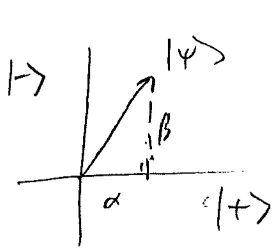
If measure $S_z$ of $|+\rangle$ will always get $\frac{\hbar}{2}$

$|-\rangle$ " " " $-\frac{\hbar}{2}$

## Indeterminate states

Even though $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are the only possible results of a measurement of $S_z$, $|+\rangle$ and $|-\rangle$ are not the only possible states

Hilbert space contains also $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ $\quad \alpha\beta \in \mathbb{C}$




← picture assumes $\alpha, \beta \in \mathbb{R}$, but $\alpha, \beta \in \mathbb{C}$ in general

We now assume that $|\alpha|^2 + |\beta|^2 = 1$ ...

Suppose otherwise: $|\alpha|^2 + |\beta|^2 = N$.

Then define $\alpha' = \frac{\alpha}{\sqrt{N}}$, $\beta' = \frac{\beta}{\sqrt{N}}$ $\Rightarrow |\alpha'|^2 + |\beta'|^2 = 1$

Let $|\psi'\rangle = \alpha'|+\rangle + \beta'|-\rangle = \frac{1}{\sqrt{N}}|\psi\rangle$. ← call this a "normalized state"

Since $|\psi'\rangle$ & $|\psi\rangle$ describe same state, use $|\psi'\rangle$.

$\boxed{\text{Probabilities}}$

What happens if measure $S_z$ of state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$?

<u>Must</u> get either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

In fact, <u>can</u> get either
prob. of result $\frac{\hbar}{2}$ is given by $|\alpha|^2$
prob of result $-\frac{\hbar}{2}$ is given by $|\beta|^2$

Check that this makes sense:

- since $|\alpha|^2 + |\beta|^2 = 1$, probabilities add to 1

- Suppose $\beta = 0$. Then $|\alpha|^2 = 1 \Rightarrow \alpha = e^{i\theta}$
  $|\psi\rangle = e^{i\theta}|+\rangle$ which is equivalent to determinate state $|+\rangle$
  Then prob of $\frac{\hbar}{2}$ is $|\alpha|^2 = 1$
  and prob of $-\frac{\hbar}{2}$ is $|\beta|^2 = 0$. ✓

- vice versa if $\alpha = 0$

Thus, given a state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$
we can predict the probabilities of various results

but for any individual measurement, we cannot predict the result

In QM, measurement is an <u>inherently random process</u>

This is a <u>radical</u> departure from classical mechanics.

Usually in physics randomness arises from our ignorance of the complete specification of a system

E.g. coin flip is random only because we do not know the exact details of the flip.

If we knew precisely the initial conditions, we could predict the results of a flip.

Newton's laws are deterministic

[quote from Laplace]

In other words Given laws of physics, plus a complete and exact specification of initial positions and velocities of all the particles, we can predict their subsequent motion. In practice, we cannot know the initial condition exactly (+ in a chaotic system, even a slight variation of initial condition can lead to vastly different outcomes ⇒ "sensitive dependence on initial conditions")

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Quantum mechanics relinquishes this determinism in a fundamental way, because we assume that $|\psi\rangle$ contains all information about a state, so all we can know are probabilities. Result of an individual measurement is not only unknown, but unknowable

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In classical physics, given the ~~same~~ initial condition the same result inevitably follows

In QM measurement of the same initial state $|\psi\rangle$ can yield different results in different runs of the expt

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past could be present before its eyes.

— Pierre Simon Laplace, A Philosophical Essay on Probabilities

This randomness is hard to swallow

Even if we cannot predict the outcome,

we feel as though there must be some reasons for it.

How does nature decide?
How can it be truly random?
(or is this just some prejudice?)

Einstein felt strongly that there must be some underlying mechanism: "God does not play dice"

On the other hand, Bohr felt that nature does not have to behave according to our prejudice
"Stop telling God what to do!"

Einstein did not disagree w/ the predictions of QM, he felt that it was incomplete.
Perhaps when we describe a state by $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$
we do not know everything about the state
Perhaps there are additional variables that are hidden from us $\Rightarrow$ "hidden variable theories"

E.g. given an ensemble of $N$ apparently identical states $|\psi\rangle$, there is something that distinguishes them:
$N_+$ of them are actually $|\psi,+\rangle \Rightarrow \frac{\hbar}{2}$
$N_-$ of them are actually $|\psi,-\rangle \Rightarrow -\frac{\hbar}{2}$

so prob of getting $\frac{\hbar}{2}$ is $\frac{N_+}{N}$
$-\frac{\hbar}{2}$ " $\frac{N_-}{N}$

(we'll discuss such hidden variable theories later)

"God does not play dice."

Albert Einstein

"Einstein, stop telling God what to do."

Niels Bohr

*A philosopher once said, "It is necessary for the very existence of science that the same conditions always produce the same results." Well, they don't!*

RICHARD FEYNMAN, *The Character of Physical Law* (1965)

Free will has always seemed hard to reconcile w/ physics.

If the future is fully accounted for by initial conditions then our actions were determined long before we were born.

Yet I believe we all psychologically feel as though we can make choices

Moreover, we praise & blame others for their actions & hold them legally responsible which ~~doesn't~~ doesn't make sense if they don't actually have a choice

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There once was a man who said Damn
It grieves me to think that I am
predestined to move
in a circumscribed groove
In fact, not a bus, but a tram

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Some have claimed that QM restores free will but I don't believe so.
Is randomness any better?
Classical mechanics say I was determined to rob the bank
QM say I have a 60% chance of robbing it
& 40% not
but the outcome is still not in my control.

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My conclusion is that you should expect physics to explain everything in our experience (at least not yet)

What about multiple measurements?

Given a state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

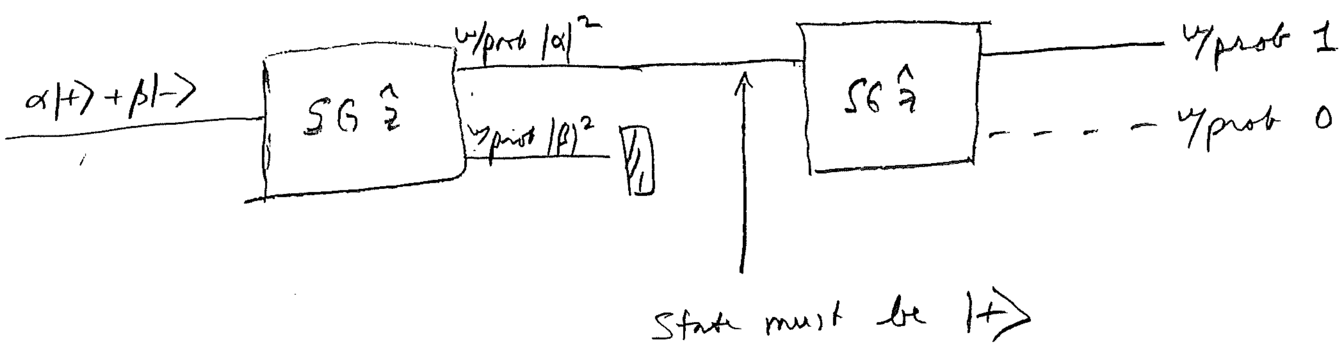
measure $S_z$ and get $\frac{\hbar}{2}$ w/ prob. $|\alpha|^2$

measure $S_z$ again immediately afterward,
we expect to get same result $\frac{\hbar}{2}$ w/ prob 1

This would not be the case if the state were still
$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

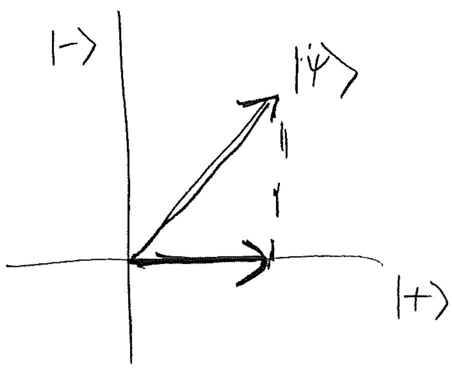
$\therefore$ first measurement must necessarily affect the state

To get $\frac{\hbar}{2}$ w/ prob 1
the state before the 2nd measurement must have been $|+\rangle$,
(or $e^{i\theta}|+\rangle$)



The first measurement changes the state

$$\alpha|+\rangle + \beta|-\rangle \longrightarrow |+\rangle$$




Projection postulate:
measurement projects the
state onto a determinate state

Reduction or collapse of the wavefunction

"Copenhagen interpretation of QM"
developed by Bohr, Heisenberg + others

There are now two sets of rules

① Schrödinger equation describes smooth, continuous
causal evolution of state $|\psi\rangle$
before + after measurement

② measurement of an observable induces
sudden, discontinuous, random collapse of $|\psi\rangle$
onto a determinate state of the observable

Copenhagen interpretation gives no explanation for
how collapse occurs other than to say it happens;
can't say what "causes" it because its not causal
can't say why state is projected onto one determinate
state rather than another, because it's random

Do we have to give up on the unity of physics?

How do we reconcile these two processes?

Problem of interpretation of QM
(on which much ink has been spilled)
No consensus has emerged

Isn't a measuring device just a complicated quantum system? Why should it be described by different rules? What makes it special?

Bohr: distinction between microscopic quantum process + classical measurement apparatus

But where draw the line? Big vs small?

Schrödinger cat: a quantum cat should be able to exist in a superposition of states

But Schrödinger did not believe this.
He was trying to ridicule the idea of collapse

An even more serious problem of collapse is that it is nonlocal
Shown by EPR in paper in 1936. We'll return to this later

Hidden variables do not require collapse
A set of N states $|\psi\rangle$ really consists of two populations
$N_1$ of $|\psi,+\rangle$ and $N_2$ of $|\psi,-\rangle$
The first measurement simply determines which set a state really belongs to.
The second measurement simply confirms this.

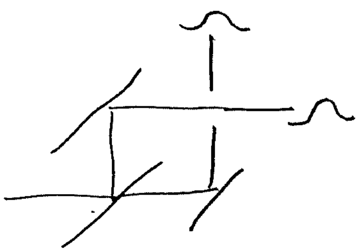
But conflict w/ experiment

## Nonlocality of collapse

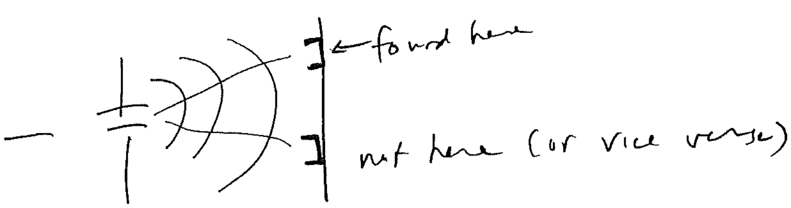
If we measure pcl it can only be in one place

How does measurement apparatus at $y$ know that the pcl has been found at $x$ before enough time has elapsed for signal to travel fm $x \to y$

one-dimensional scattering



Interference



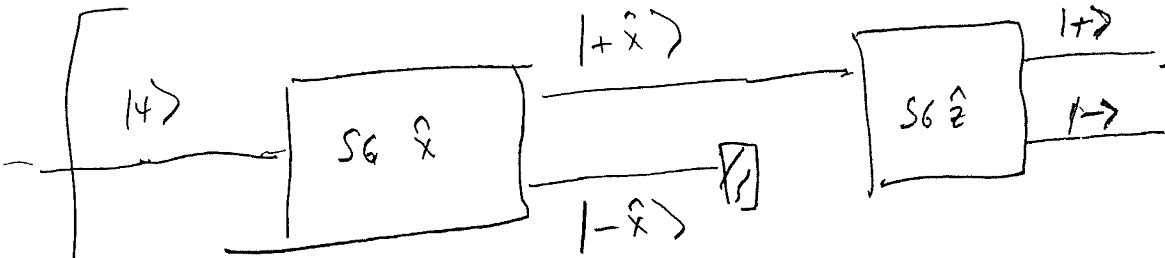

diffraction


Stern-Gerlach

Wait until 8-10

12b

4-




Let $|\pm\hat{x}\rangle$ be determinate states of $S_x$ w/ results $\pm\frac{\hbar}{2}$

[For completeness, we should write $|\pm\rangle = |\pm\hat{z}\rangle$ but we won't]

Second measurement suggests $|+\hat{x}\rangle = \alpha|+\rangle + \beta|-\rangle$

with $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$.

(Later we'll discover that $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$
$|-\hat{x}\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$)

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$\therefore$ determinate states of $S_x$ are not the same as determinate states of $S_z$

$\Rightarrow S_x + S_z$ are incompatible

perhaps not necessary here

explained on 8-10

[mathematicians spend a lot of time trying to prove this, but we will simply assume it is true w/o proving it]

## Completeness

Assume: the determinate states of an observable A constitute a complete basis for the space of all states

ie **any** state can be written as a linear combination of basis states

Suppose observable A has a finite spectrum $\{a_n\}$
(+ corresponding independent determinate states $|a_n\rangle$ $n=1,\dots d$
(eg $d=2$ for $S_z$, viz $|+\rangle$ and $|-\rangle$)
then the most general state is

$$|\psi\rangle = \sum_{n=1}^{d} \psi_n |a_n\rangle$$

→ can write as a column vector $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}$

$d$ = dimension of state space
$\psi_n$ = coeff of $|\psi\rangle$ in A-basis
= probability amplitude for measuring $a_n$

Probability that a measurement of A for state $|\psi\rangle$ will yield $a_n$ is $|\psi_n|^2$

Require $\sum_{n=1}^{d} |\psi_n|^2 = 1$. (Otherwise, normalize)

Expectation value

$\langle A \rangle$ = expectation value of $A$ for the state $|\psi\rangle$

= mean value (weighted by probabilities) of results of measurement of $A$

$$= \sum |\psi_n|^2 a_n$$

Uncertainty

$\Delta A$ = uncertainty in $A$ for the state $|\psi\rangle$

= root mean square deviation from $\langle A \rangle$

- results of measurement: $a_n$
- deviation from the mean of each measurement: $a_n - \langle A \rangle$

- weighted avg of square deviation $= \sum |\psi_n|^2 (a_n - \langle A \rangle)^2$

$$= \sum |\psi_n|^2 (a_n^2 - 2\langle A \rangle a_n - \langle A \rangle^2)$$

$$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle - \langle A \rangle^2$$

- mean square deviation $= \langle A^2 \rangle - \langle A \rangle^2$
- rms deviation $= \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

For a determinate state, $\Delta A = 0$ because measurement always gives one result $a_n$ so deviation vanishes.

[Converse is also true.] ← this is implied