rev'd after 2023

Position space

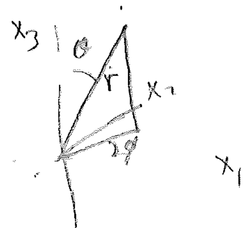
Write operators in position space

$$p_i \to \frac{\hbar}{i}\frac{\partial}{\partial x_i} \qquad \text{ie} \qquad \vec{p} \to \frac{\hbar}{i}\vec{\nabla}$$

$$H = \sum \frac{p_i^2}{2m} + V(r) \longrightarrow -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$L_i = \epsilon_{ijk}\, x_j\, p_k \longrightarrow \frac{\hbar}{i}\epsilon_{ijk}\, x_j \frac{\partial}{\partial x_k}$$

Recall spherical coordinates



$$\begin{cases} x_3 = r\cos\theta \\ x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \end{cases}$$

$$L_3 = \frac{\hbar}{i}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

I claim: $L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

<u>Proof</u> (chain rule): $\frac{\partial}{\partial\phi} \quad \frac{\partial x_1}{\partial\phi}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial\phi}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial\phi}\frac{\partial}{\partial x_3}$

$$= \underbrace{-r\sin\theta\sin\phi}_{x_2}\frac{\partial}{\partial x_1} + \underbrace{r\sin\theta\cos\phi}_{x_1}\frac{\partial}{\partial x_2} + 0$$

$$= x_1\frac{\partial}{\partial x_2} \quad x_2\frac{\partial}{\partial x_1}$$

[Alternatively. $U\psi = e^{-\frac{i\alpha L_3}{\hbar}}\psi = e^{-\alpha\frac{\partial}{\partial\phi}}\psi = \left(1 - \alpha\frac{\partial}{\partial\phi}\psi + \frac{1}{2}\alpha^2\frac{\partial^2\psi}{\partial\phi^2} \cdots\right)$

$= \psi(r, \theta, \phi - \alpha) \Rightarrow$ rotate ψ through angle α]

[HW]: show that

$$L_1 = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right)$$

$$L_2 = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right)$$

This implies

$$L_\pm = L_1 \pm i L_2 = i\hbar\left(\underbrace{[\sin\phi \mp i\cos\phi]}_{\mp i e^{\pm i\phi}} \frac{\partial}{\partial\theta} + \cot\theta \underbrace{[\cos\phi \pm i \sin\phi]}_{e^{\pm i\phi}} \frac{\partial}{\partial\phi}\right)$$

$$\boxed{L_\pm = \hbar e^{\pm i\phi}\left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right)}$$

Save this for later

Position state eigenstates of $\{H, L^2, L_3\}$

$$u(r, \theta, \phi) = \langle \vec{x} \,|\, E, l, m \rangle$$

↑ should be labelled by E, l, m

Eigenvalue eqns are just linear P.D.E.'s

Try separable ansatz $u(r,\theta,\phi) = R(r) \underbrace{Y_{lm}(\theta,\phi)}_{g(\theta)\,\boxed{f(\phi)}}$

↖ don't use $h(\phi)$!

$$L_3 |E, l, m\rangle = m\hbar\, |E, l, m\rangle$$

$$\frac{\hbar}{i}\frac{\partial}{\partial \phi} u(r,\theta,\phi) = m\hbar\, u$$

$$\frac{\hbar}{i}\frac{df}{d\phi} = m\hbar f$$

$$\frac{df}{f} = i m\, d\phi$$

$$\ln f = i m \phi + \text{const}$$

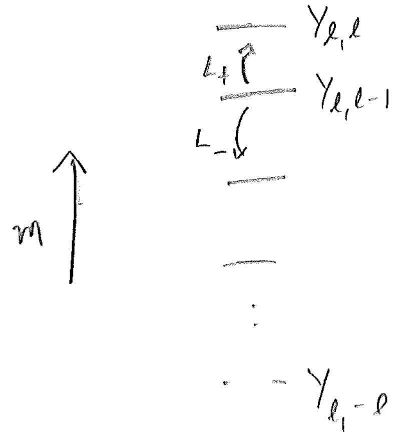
$$f = (\text{const})\, e^{im\phi} \qquad \Rightarrow f(\phi + 2\pi) = e^{2\pi i m} f(\phi)$$

We know already that for any type of angular momentum
m = integer or half-integer.

But for *orbital* angular momentum, m must be *integer* in order that $f(\phi + 2\pi) = f(\phi)$, ie single valued function of position
(otherwise $f(\phi + 2\pi) = -f(\phi)$)

$$\Rightarrow Y_{lm}(\theta, \phi) = g(\theta)\, e^{im\phi}, \qquad m = -l, \dots, l$$

Recall $L_\pm = \hbar e^{\pm i\phi}\left(\pm \frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$



$$L_+ Y_{\ell\ell} = 0$$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right) g(\theta)e^{i\ell\phi} = 0$$

$$\frac{dg}{d\theta} + i\cot\theta\,(i\ell)\,g = 0$$

$$\frac{dg}{d\theta} = -g\,\ell\cot\theta$$

$$\frac{dg}{g} = -\ell\,\frac{\cos\theta\,d\theta}{\sin\theta}$$

$$\ln g = -\ell\ln(\sin\theta) + \text{const}$$

$$g = C(\sin\theta)^\ell$$

$$Y_{\ell\ell}(\theta,\phi) = C(\sin\theta)^\ell e^{i\ell\phi}$$

[determine C later]

Recall $L_\pm Y_{\ell m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\, \hbar\, Y_{\ell, m\pm 1}$

$$Y_{\ell,\ell-1} = \frac{1}{\sqrt{2\ell}} \frac{L_-}{\hbar} Y_{\ell,\ell}$$

$$= \frac{e^{-i\phi}}{\sqrt{2\ell}}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right) C(\sin\theta)^\ell e^{i\ell\phi}$$

$$= \frac{e^{-i\phi}C}{\sqrt{2\ell}}\left(-\ell(\sin\theta)^{\ell-1}\cos\theta - \ell(\cos\theta)(\sin\theta)^{\ell-1}\right)e^{i\ell\phi}$$

$$Y_{\ell,\ell-1} = -\sqrt{2\ell}\; C\; (\sin\theta)^{\ell-1}(\cos\theta)\, e^{i(\ell-1)\phi}$$

etc

$Y_{\ell m}(\theta, \phi)$ are normalized by

$$\int |Y_{\ell m}(\theta, \phi)|^2 \underbrace{d\Omega}_{\text{solid angle measure } = \sin\theta \, d\theta \, d\phi} = 1$$

For example

$$\int |Y_{\ell\ell}(\theta, \phi)|^2 \sin\theta \, d\theta \, d\phi$$

$$= |C|^2 \underbrace{\int_0^{\pi} (\sin\theta)^{2\ell+1} d\theta}_{\frac{2(2\ell)!!}{(2\ell+1)!!}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

Recall $n!! = n(n-2)(n-4)\dots$

$$\Rightarrow C = \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!} \frac{1}{4\pi}} \cdot \underset{\nwarrow \text{standard convention}}{(-1)^{\ell}}$$

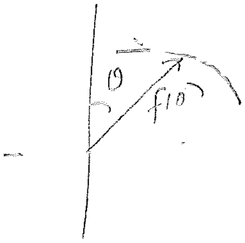
$$\left[\int_0^{\pi} \sin^{2\ell+1}\theta \, d\theta = \int_{-1}^{1} \underbrace{(1-x^2)^{\ell}}_{\sum_{m=0}^{\ell} (-)^m x^{2m} \cdot \frac{\ell!}{m!(\ell-m)!}} dx = 2 \sum_{m=0}^{\ell} (-1)^m \frac{\ell!}{m!(\ell-m)!} \frac{1}{(2m+1)} = \frac{2(2\ell)!!}{(2\ell+1)!!} \right]$$

	ℓ	
S. state	0	$Y_{00} = \frac{1}{\sqrt{4\pi}}$
P-state	1	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$ $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$ $Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$
D-state	2	$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$ $Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}$ $Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$ $Y_{2,-1} = +\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{-i\phi}$ $Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{-2i\phi}$

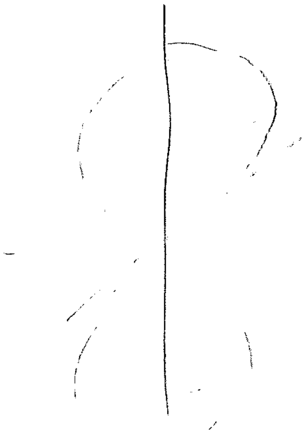
$$Y_{\ell,-m} = (-1)^m \, Y^*_{\ell,m}$$

[cf Griffiths (2e) 139]

visualize using polar ~~plot~~ plots
(explain



$|Y_{10}|^2 \sim \cos^2\theta$



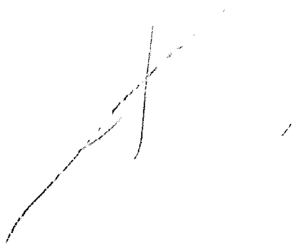
← "p_z"

$|Y_{11}|^2 \sim \sin^2\theta$

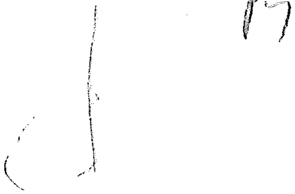
$\sim |Y_{1,-1}|^2$

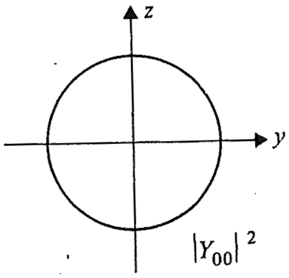
$Y_{1,1} - Y_{1,-1} \sim \sin\theta\,(e^{i\phi} + e^{-i\phi}) \sim 2\sin\theta\cos\phi$

$|Y_{11} - Y_{1,-1}|^2 \sim \sin^2\theta\cos^2\phi$

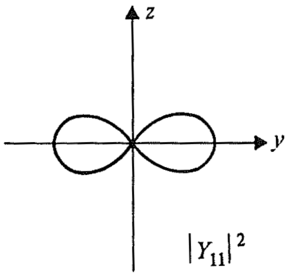


$|Y_{11} - Y_{1,-1}|^2 \sim \sin\theta\,\sin\phi$

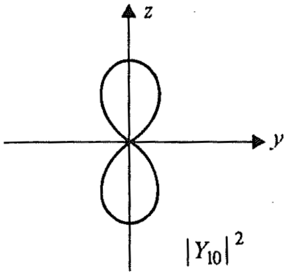




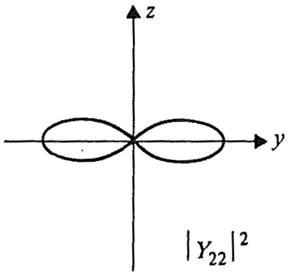
$|Y_{00}|^2$



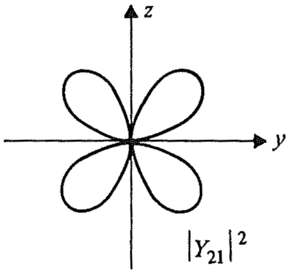
$|Y_{11}|^2$



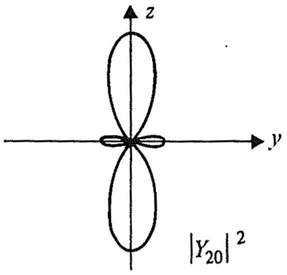
$|Y_{10}|^2$



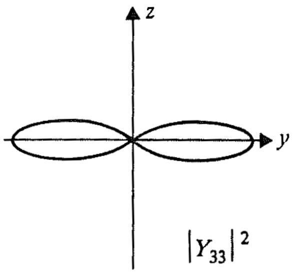
$|Y_{22}|^2$



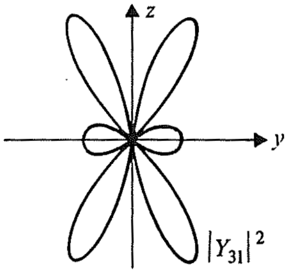
$|Y_{21}|^2$



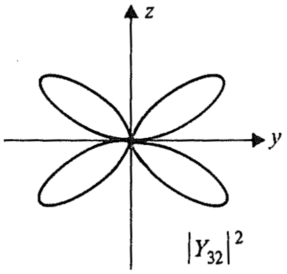
$|Y_{20}|^2$



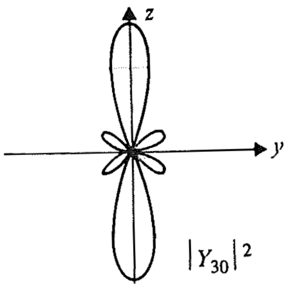
$|Y_{33}|^2$



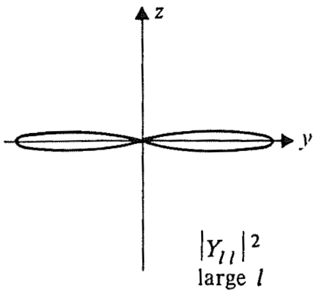
$|Y_{31}|^2$



$|Y_{32}|^2$



$|Y_{30}|^2$



$|Y_{l\,l}|^2$
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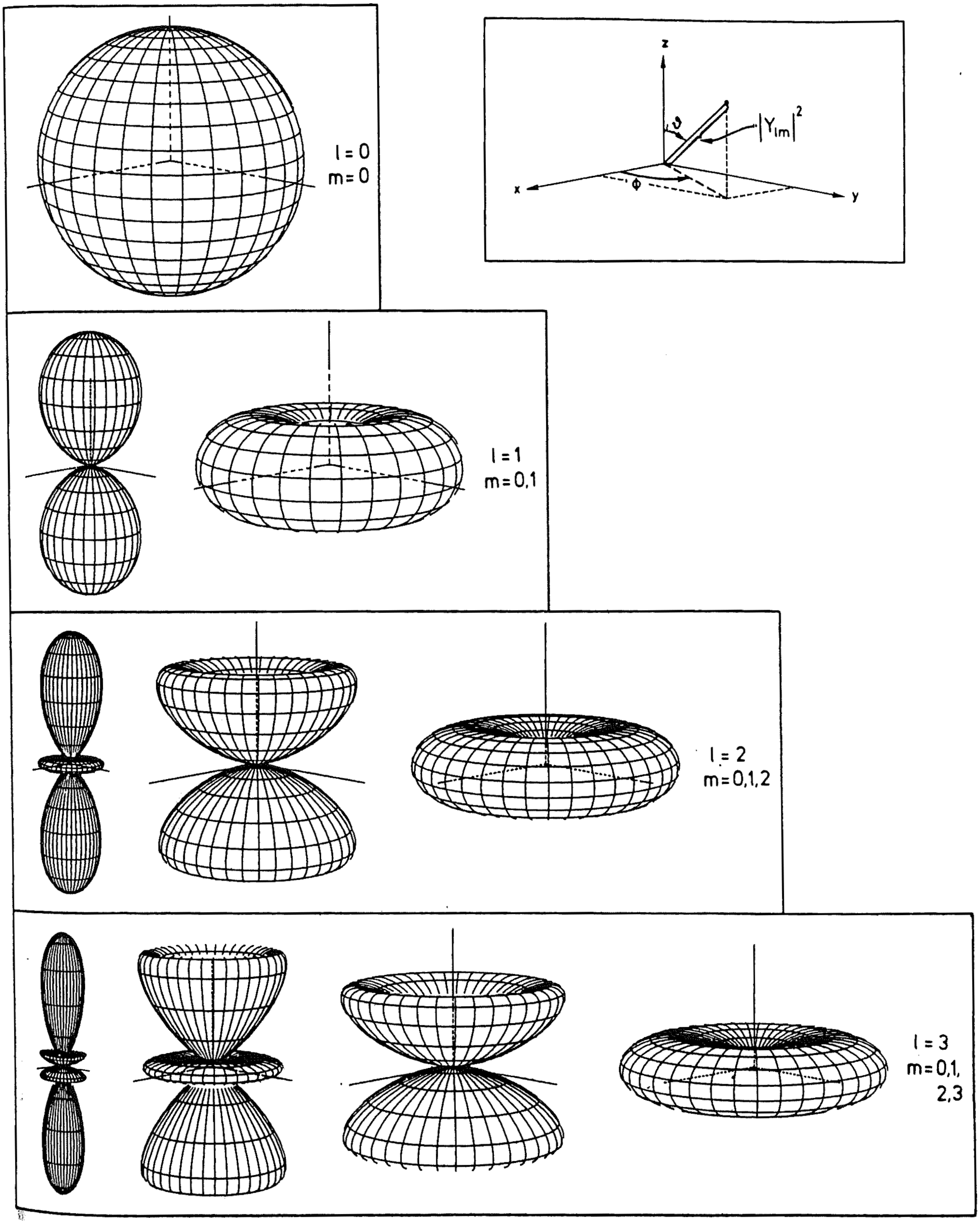
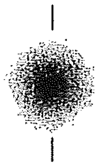
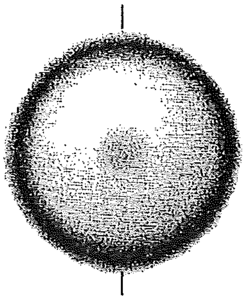
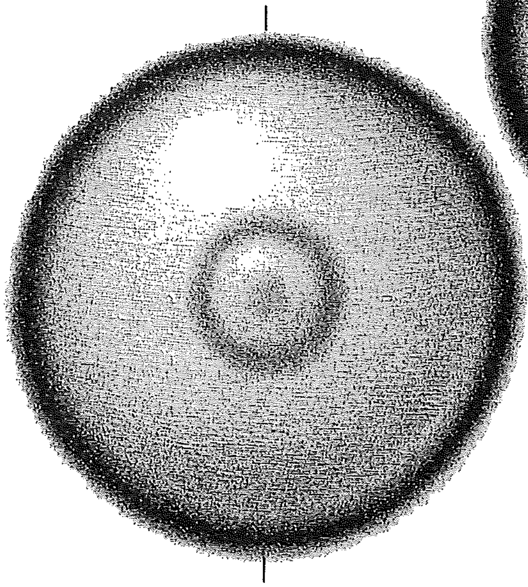




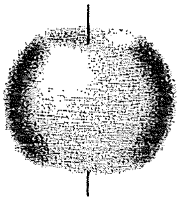
$n = 1,\ l = m_l = 0$



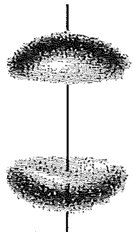
$n = 2,\ l = m_l = 0$



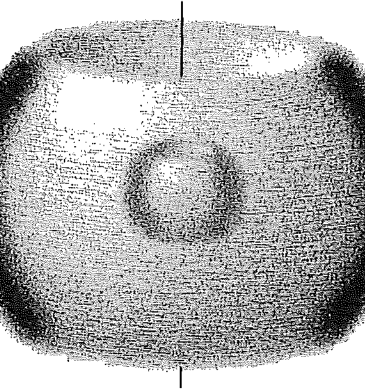
$n = 3,\ l = m_l = 0$



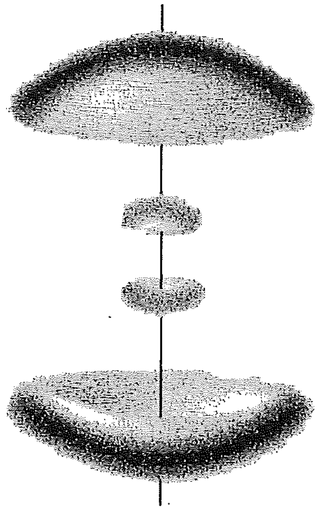
$n = 2,\ l = 1,\ m_l = \pm 1$



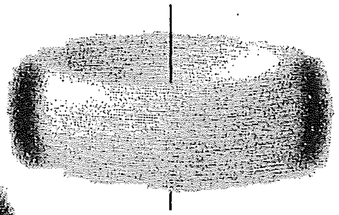
$n = 2,\ l = 1,\ m_l = 0$



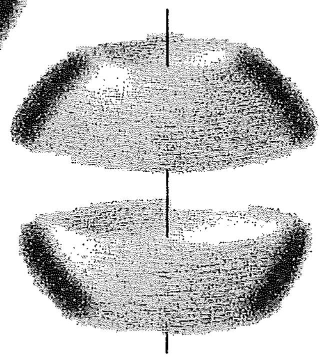
$n = 3,\ l = 1,\ m_l = \pm 1$



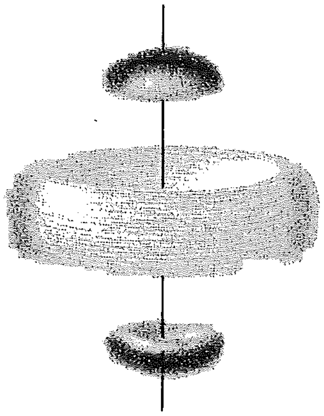
$n = 3,\ l = 1,\ m_l = 0$



$n = 3,\ l = 2,\ m_l = \pm 2$



$n = 3,\ l = 2,\ m_l = \pm 1$



$n = 3,\ l = 2,\ m_l = 0$