revd after 2023

Angular momentum

Recall rotations generated by angular momentum operator $\vec{J}$

$$U = e^{-\frac{i\alpha \hat{n}\cdot\vec{J}}{\hbar}}$$ for rotation through α about $\hat{n}$

In HW, you showed this implies

$[J_1, J_2] = i\hbar J_3$ plus cyclic permutations

one can write all 3 eqns as

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k$$

ϵ_{ijk} = Levi-Civita tensor $\Rightarrow$ set of 27 constants,

antisymmetric under exchange of any two indices

Define $\epsilon_{123} = 1$

Then $\epsilon_{213} = -\epsilon_{123} = -1$

$\epsilon_{132} = -1$

$\epsilon_{312} = -1$

$\epsilon_{321} = 1$

$\epsilon_{213} = 1$

$\epsilon_{112} = -\epsilon_{112} \Rightarrow \epsilon_{112} = 0$ etc.

$$[J_1, J_2] = i\hbar(\underbrace{\epsilon_{121}}_{0} J_1 + \underbrace{\epsilon_{122}}_{0} J_2 + \underbrace{\epsilon_{123}}_{1} J_3) = i\hbar J_3$$

Orbital angular momentum

$$\vec{L} = \vec{x} \times \vec{p}$$

i.e. $L_3 = x_1 p_2 - x_2 p_1$

Can also write these as

$$L_i = \sum_j \sum_k \epsilon_{ijk} x_j p_k$$

Useful identity

$$\sum_{k=1}^{3} \epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

[Proof in words: ij & lm different than k, so 2 possibilities]

Quantize.

$$\hat{L}_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \hat{x}_j \hat{p}_k \quad \xRightarrow{[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}} \quad [\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k$$

[HW] show $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ implies

$$[\hat{L}_i, \hat{x}_j] = i\hbar \sum \epsilon_{ijk} \hat{x}_k$$

$$[\hat{L}_i, \hat{p}_j] = i\hbar \sum \epsilon_{ijk} \hat{p}_k$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum \epsilon_{ijk} \hat{L}_k$$

Solutions

$$[L_i, p_j] = \epsilon_{ilm}[x_l, p_j]p_m = i\hbar\,\epsilon_{ijm}p_m$$

$$[L_i, x_j] = \epsilon_{ilm}x_l[p_m, x_j] = -i\hbar\,\epsilon_{ilj}x_l = i\hbar\,\epsilon_{ijl}x_l$$

$$[L_i, L_j] = \epsilon_{jlm}[L_i, x_l p_m]$$

$$= \epsilon_{jlm}\left([L_i, x_l]p_m + x_l[L_i, p_m]\right)$$

$$= i\hbar\,\underset{\uparrow}{\epsilon_{jlm}}\left(\epsilon_{iln}x_n p_m + \epsilon_{imn}x_l p_n\right)$$

$$= i\hbar\left([\delta_{ij}\delta_{mn} - \delta_{jn}\delta_{im}]x_n p_m + [\delta_{jn}\delta_{li} - \delta_{ji}\delta_{mn}]x_l p_n\right)$$

$$= i\hbar\left(-x_j p_i + x_i p_j\right)$$

$$= i\hbar\,\epsilon_{ijk}L_k$$

where $\epsilon_{ijk}L_k = \epsilon_{ijk}\epsilon_{klm}x_l p_m$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})x_l p_m$$

$$= x_i p_j - x_j p_i$$

Useful to use Einstein summation convention:
repeated indices are summed over

$$[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k$$

$$L_i = \epsilon_{ijk}\, x_j\, p_k$$

Define L^2 $L_m L_m$ (ie $\sum_{m=1}^{3} L_m^2$)

Consider $$[L_i, L^2] = \underbrace{[L_i, L_m]}_{i\hbar\epsilon_{imj} L_j} L_m + L_m \underbrace{[L_i, L_m]}_{i\hbar\epsilon_{imj} L_j}$$

$$= i\hbar\,\epsilon_{imj} \underbrace{(L_j L_m + L_m L_j)}_{\text{vanishes because antisym. \& sym.}}$$

$$A_{ij} B_{ij} = (-A_{ji})(B_{ji}) \underset{\text{relabel indices}}{=} -A_{ij}B_{ij}$$

$$\Rightarrow 2A_{ij}B_{ij} = 0$$

L^2 commutes with each component of L_i
(& so also w/ $L_\pm$)

Recall $$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$
$$L_3 |l, m\rangle = \hbar m |l, m\rangle$$

$$H = \sum \frac{p_i^2}{2m} + V(x_i)$$

Assume spherical symmetry, i.e. $V = V(r)$ (not θ, ϕ)

Then $\vec{F} = -\vec{\nabla} V \sim \hat{r}$ force in radial direction only
("central potential")

We'll show that L_i commutes w/ H for a central potential

Consider $[L_i, p_m p_m] = \epsilon_{ijk} [x_j p_k, p_m p_m]$

$$= \epsilon_{ijk} [x_j, p_m p_m] p_k$$

$$= \epsilon_{ijk} \big(\underbrace{[x_j, p_m]}_{i\hbar\delta_{jm}} p_m + p_m [x_j, p_m] \big) p_k$$

$$= 2i\hbar \, \epsilon_{ijk} \delta_{jm} p_m p_k$$

$$= 2i\hbar \, \epsilon_{imk} p_m p_k = 0$$

Consider $[L_i, V] = \epsilon_{ijk} [x_j p_k, V]$

$$= \epsilon_{ijk} x_j \underbrace{[p_k, V]}_{-i\hbar \frac{\partial V}{\partial x_k}} = -i\hbar \underbrace{\epsilon_{ijk} x_j x_k}_{0} \frac{1}{r}\frac{dV}{dr}$$

central: $\frac{\partial V}{\partial x_k} = \frac{dV}{dr}\frac{\partial r}{\partial x_k} = \frac{dV}{dr}\frac{x_k}{r}$

$\Rightarrow [L_i, H] = 0$

Also $[L^2, H] = L_m [L_m, H] + [L_m, H] L_m = 0$

$\therefore \{H, L^2, L_3\}$ form a set of mutually commuting operators

Goal: find mutual eigenstates $|E, l, m\rangle$:

$$H|E,l,m\rangle = E\,|E,l,m\rangle$$
$$L^2|E,l,m\rangle = \hbar^2 l(l+1)\,|E,l,m\rangle$$
$$L_3|E,l,m\rangle = \hbar m\,|E,l,m\rangle$$

[Not necessary to present this.]

more general argument:

H is invariant under rotations because
$V(r)$ only depends on r and ∇^2 is a scalar operator

Let $H|E\rangle = E|E\rangle$

Then $U|E\rangle$ is also an eig-state of same energy E
(because rotating the state will produce another state with same energy)

so $H(U|E\rangle) = E(U|E\rangle)$

$HU|E\rangle = UE|E\rangle = UH|E\rangle$

$(HU - UH)|E\rangle = 0$

$[H, U]|E\rangle = 0$

Because energy eig-states are a complete basis
$[H, U]|\psi\rangle = 0$ for any $|\psi\rangle$

Because this holds for any $|\psi\rangle$

$$[H, U] = 0$$

$$U = e^{-\frac{i\alpha \hat{n}\cdot\vec{J}}{\hbar}} = 1 - \frac{i\alpha}{\hbar}\hat{n}\cdot\vec{J} + \ldots$$

$$\Rightarrow [H, \hat{n}\cdot\vec{J}] = 0$$

$$[H, \vec{J}_i] = 0$$

Particle w/o spin
$$[H, \vec{L}_i] = 0.$$