Energy eigenstate

$$H\,|E\rangle = E\,|E\rangle$$

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2\right) + V(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

In position space

$$\langle \vec{x}\,|E\rangle \equiv u_E(\vec{x})$$

$$\langle \vec{x}\,|H|E\rangle = H_{pos}\,\langle \vec{x}\,|E\rangle = E\,\langle \vec{x}\,|E\rangle$$

$$H_{pos} = -\frac{\hbar^2}{2m}\underbrace{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)}_{\nabla^2} + V(x_1, x_2, x_3)$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right] u_E(\vec{x}) = E\,u_E(\vec{x})$$

3d t.i.S.e

Consider a 3d potential of the special form

$$V(x_1, x_2, x_3) = \tilde{V}(x_1) + \tilde{V}(x_2) + \tilde{V}(x_3)$$ → symmetric under $x_1 \leftrightarrow x_2$ (or any perm of x_1, x_2, x_3)

This includes 1) particle in a cube

2) isotropic harmonic oscillator

$$V(\vec{x}) = \frac{1}{2} k \, \vec{x} \cdot \vec{x}$$

Then

$$\sum_{i=1}^{3} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \tilde{V}(x_i) \right] u(\vec{x}) = E \, u(\vec{x})$$

Try a separable ansatz. $u(\vec{x}) = u^{(1)}(x_1)\, u^{(2)}(x_2)\, u^{(3)}(x_3)$

$$\sum \underbrace{\frac{1}{u^{(i)}} \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \tilde{V}(x_i) \right] u^{(i)}(x_i)}_{E^{(i)}} = E$$

$$E = E^{(1)} + E^{(2)} + E^{(3)}$$

$$-\frac{\hbar^2}{2m} \frac{d^2 u^{(i)}}{dx^2} + \tilde{V}(x) \, u^{(i)} = E^{(i)} u^{(i)}$$ ← 1d eqn, same eqn for all i

Let $u_n(x)$ be solutions of this eqn of eigenenergies E_n

For each i, choose one of these solutions

$$u(x_1, x_2, x_3) = u_{n_1}(x_1)\, u_{n_2}(x_2)\, u_{n_3}(x_3)$$

$$E = E_{n_1} + E_{n_2} + E_{n_3}$$

		E	degeneracy
gnd state	$u_1(x_1)\,u_1(x_2)\,u_1(x_3)$	$3E_1$	1
1st excited	$u_2(x_1)\,u_1(x_2)\,u_1(x_3)$ $u_1(x_1)\,u_2(x_2)\,u_1(x_3)$ $u_1(x_1)\,u_1(x_1)\,u_2(x_3)$	$2E_1+E_2$	3

	E	degeneracy
	E_1+2E_2	3
	$2E_1+E_3$	3
not nec. in this order		
	$E_1+E_2+E_3$	6

Bound states in 1d. are always nondegenerate

In >1d, there are often degeneracies
especially when potential has symmetry

[e.g.