scm 12/1(2023)

3 dimensions

position operators $\hat{X}, \hat{Y}, \hat{Z}$ or $\hat{X}_i$ $(i=1,2,3)$

momentum operators $\hat{P}_x, \hat{P}_y, \hat{P}_z$ or $\hat{P}_i$ $(i=1,2,3)$

$$[\hat{X}_i, \hat{X}_j] = 0$$
$$[\hat{P}_i, \hat{P}_j] = 0$$

} necessary to do this in order to have them compute $[\hat{L}_i, \hat{L}_j]$ later

$$[\hat{X}_i, \hat{P}_j] = i\hbar\,\delta_{ij}$$

To prove these, use position space operators

$$\hat{X}_i \to x_i \qquad \Rightarrow [x_i, x_j]\psi = 0$$

$$\hat{P}_i \to \frac{\hbar}{i}\frac{\partial}{\partial x_i} \qquad \Rightarrow \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]\psi = 0$$

$$[\hat{X}_i, \hat{P}_j]|\psi\rangle \to x_i \frac{\hbar}{i}\frac{\partial}{\partial x_j}\psi - \frac{\hbar}{i}\frac{\partial}{\partial x_j}(x_i\psi)$$

$$= i\hbar \frac{\partial x_i}{\partial x_j}\psi = i\hbar\,\delta_{ij}\psi$$

use these to derive uncertainty relations

$$\Delta x \; \Delta p_x \geq \frac{\hbar}{2}$$
$$\Delta y \; \Delta p_y \geq \frac{\hbar}{2}$$
$$\Delta z \; \Delta p_z \geq \frac{\hbar}{2}$$

but no constraints on $\Delta x \; \Delta p_y$ for example

(Also $\underset{\sim}{\Delta t}\; \Delta E \geq \frac{\hbar}{2}$)

τ = decay time of a state

Let $|x_1, x_2, x_3\rangle = |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle$

$\hat{X}_1$ acts as $\hat{X} \otimes \mathbb{1} \otimes \mathbb{1}$ etc. so $\hat{X}_i |x_1, x_2, x_3\rangle = x_i |x_1, x_2, x_3\rangle$

= simultaneous eigenstate of $\hat{X}_1, \hat{X}_2, \hat{X}_3$

Convenient notation:

$|\vec{x}\rangle = |x_1, x_2, x_3\rangle$ = particle "localized" at $\vec{x}$

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orthonormality $\langle \vec{x}' | \vec{x} \rangle = \langle x_1' | x_1 \rangle \langle x_2' | x_2 \rangle \langle x_3' | x_3 \rangle = \underbrace{\prod_{i=1}^{3} \delta(x_i - x_i')}_{\equiv \delta^{(3)}(\vec{x} - \vec{x}')}$

$\int \underbrace{d^3x}_{dx_1 dx_2 dx_3} f(\vec{x}) \, \delta^{(3)}(\vec{x} - \vec{x}_0) = f(\vec{x}_0)$

completeness: $\int d^3x \, |\vec{x}\rangle \langle \vec{x}| = \mathbb{1}$

$|\psi\rangle = \int d^3x \, |\vec{x}\rangle \underbrace{\langle \vec{x} | \psi \rangle}_{\equiv \psi(\vec{x})} $ $= \psi(x_1, x_2, x_3)$ wavefunction in 3d.

Similarly for momentum $|\vec{p}\rangle = |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle$ = particle w/ momentum $\vec{p}$

completeness $\int d^3p \, |\vec{p}\rangle\langle\vec{p}| = 1$

$|\psi\rangle = \int d^3p \, |\vec{p}\rangle \underbrace{\langle \vec{p} | \psi \rangle}_{\phi(\vec{p})}$

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normalization $\langle \psi | \psi \rangle = \int d^3x \, \langle \psi | \vec{x} \rangle \langle \vec{x} | \psi \rangle = \int d^3x \, |\psi(\vec{x})|^2 = 1$

$= \int d^3p \, \langle \psi | \vec{p} \rangle \langle \vec{p} | \psi \rangle = \int d^3p \, |\phi(\vec{p})|^2 = 1$

$\langle \vec{x} | \vec{p} \rangle = \prod_{i=1}^{3} \langle x_i | p_i \rangle = \prod_i \frac{1}{(2\pi\hbar)^{1/2}} e^{i p_i x_i / \hbar} = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i \vec{p} \cdot \vec{x}}{\hbar}}$

$\psi(\vec{x}) = \langle \vec{x} | \psi \rangle = \int d^3p \, \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \psi \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p \, e^{\frac{i \vec{p} \cdot \vec{x}}{\hbar}} \psi(\vec{x})$

(3d Fourier transf.)