Summarize

$\langle x | x' \rangle = \delta(x-x')$ $\qquad \psi(x) = \langle x | \psi \rangle$

$\langle p | p' \rangle = \delta(p-p')$ $\qquad \phi(p) = \langle p | \psi \rangle$

$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$

How are $\psi(x)$ and $\phi(p)$ related?

$$\psi(x) = \langle x | \psi \rangle = \int dp \, \langle x | p \rangle \langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \, e^{\frac{ipx}{\hbar}} \phi(p)$$

ψ is Fourier transform of ϕ

$$\phi(p) = \langle p | \psi \rangle = \int dx \, \langle p | x \rangle \langle x | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-\frac{ipx}{\hbar}} \psi(x)$$

inverse transform

Also useful

$$\delta(x-x') = \langle x | x' \rangle = \int dp \, \langle x | p \rangle \langle p | x' \rangle$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, e^{\frac{ip(x'-x)}{\hbar}} = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, e^{\frac{ip(x-x')}{\hbar}}$$

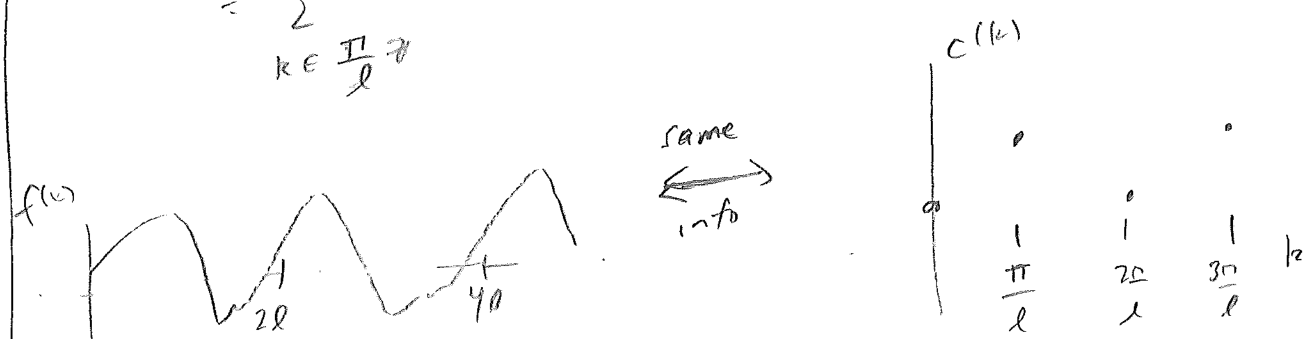
Quick review

Phys 300 I didn't do this argument

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{\ell}\right) + a_n \cos\left(\frac{n\pi x}{\ell}\right) \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n\pi x}{\ell}} \qquad \text{Let } k = \frac{n\pi}{\ell}$$

$$= \sum_{k \in \frac{\pi}{\ell}\mathbb{Z}} c(k)\, e^{ikx}$$



double period ⇒ more dense
(more information!)

$\ell \to \infty$

Gaussian momentum distribution

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$

Fourier transform

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \underbrace{\int_{-\infty}^{\infty} dp \; e^{\frac{ipx}{\hbar}} e^{-\beta(p-p_0)^2}}$$

$q = p - p_0$

$$\int_{-\infty}^{\infty} dq \; e^{\frac{i(p_0+q)x}{\hbar}} e^{-\beta q^2}$$

$$= e^{\frac{ip_0x}{\hbar}} \int_{-\infty}^{\infty} dq \; e^{-\beta q^2 + \frac{ix}{\hbar} q}$$

Recall $\int dx \, e^{-ax^2 - bx} = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{\frac{ip_0x}{\hbar}} \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{\frac{1}{4\beta}\left(-\frac{ix}{\hbar}\right)^2}$$

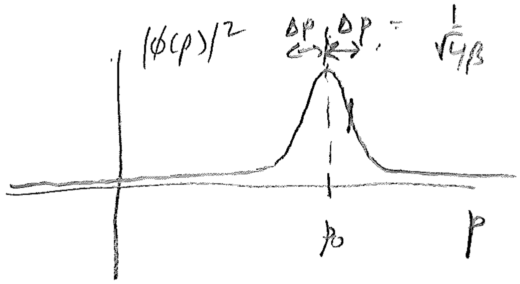
$$= \left(\frac{1}{2\pi\hbar^2\beta}\right)^{\frac{1}{4}} e^{-\frac{x^2}{4\beta\hbar^2}} e^{\frac{ip_0x}{\hbar}}$$

Define $\alpha = \frac{1}{4\beta\hbar^2}$

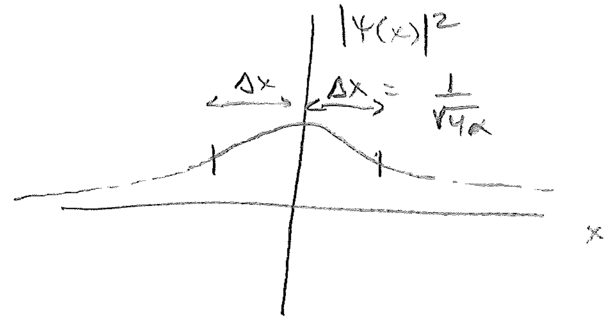
$$\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} e^{\frac{ip_0x}{\hbar}}$$

Fourier tr of a gaussian is a gaussian (times phase)

Momentum space probability density,



Position space prob density,



where $\alpha \equiv \frac{1}{4\hbar^2\beta}$

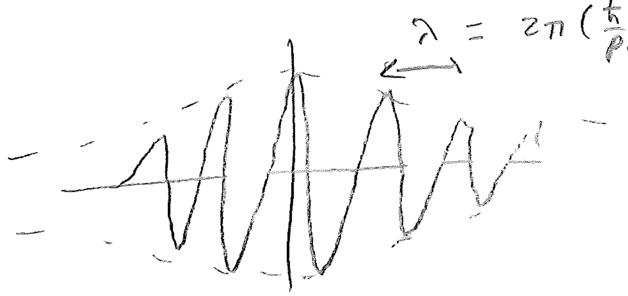
product of uncertainties $\Delta x \, \Delta p = \frac{1}{\sqrt{4\alpha}} \frac{1}{\sqrt{4\beta}} = \sqrt{\frac{\hbar^2\beta}{4\beta}} = \frac{\hbar}{2}$

Heisenberg uncertainty principle saturated for a gaussian

What about the phase in $\psi(x)$?

$$\mathrm{Re}\,\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} \cos\left(\frac{p_0 x}{\hbar}\right)$$

$\lambda = 2\pi\left(\frac{\hbar}{p_0}\right) = \frac{h}{p_0}$ = de Broglie wavelength of particle of momentum p_0



(but λ not precise because wave train is finite

How get gaussian in position space shifted by x_0?

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2} e^{-\frac{ipx_0}{\hbar}}$$

Then $\psi(x) \sim \int dp \, e^{-\beta(p-p_0)^2} e^{\frac{ip(x-x_0)}{\hbar}} = e^{-\alpha(x-x_0)^2}$

Alternative way to compute $P_{pos.}$

Start in momentum space. $P_{mom} = p$

$$\langle p \rangle = \int dp \ \phi^*(p)\, p\, \phi(p)$$

Recall $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-\frac{ipx}{\hbar}} \psi(x)$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p) \int dx \left[p\, e^{-\frac{ipx}{\hbar}} \right] \psi(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p) \int dx \left[-\frac{\hbar}{i} \frac{d}{dx} e^{-\frac{ipx}{\hbar}} \right] \psi(x)$$

IBP: $\int_{-\infty}^{\infty} u\, dv = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\, du$

$u = \psi(x)$ $\qquad dv = dx\left(-\frac{\hbar}{i}\frac{d}{dx} e^{-\frac{ipx}{\hbar}}\right)$

$du = \frac{d\psi}{dx} dx$ $\qquad v = -\frac{\hbar}{i} e^{-\frac{ipx}{\hbar}}$

$$\langle p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p) \Big[\underbrace{\psi\left(-\frac{\hbar}{i} e^{-\frac{ipx}{\hbar}}\right)\Big|_{-\infty}^{\infty}}_{0 \text{ because } \psi \underset{x\to\pm\infty}{\longrightarrow} 0} + \int_{-\infty}^{\infty} dx\, e^{-\frac{ipx}{\hbar}} \frac{\hbar}{i} \frac{d\psi}{dx} \Big]$$

$$= \int_{-\infty}^{\infty} dx\, \frac{\hbar}{i} \frac{d\psi}{dx} \underbrace{\frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p)\, e^{-\frac{ipx}{\hbar}}}_{\psi^*(x)}$$

$$\langle p \rangle = \int dx\, \psi^*(x) \underbrace{\left(\frac{\hbar}{i}\frac{d}{dx}\right)}_{P_{pos.}} \psi(x)$$

Can also use this trick to show that P is hermitian

Fourier transform of nongaussian.
(a) Calculate the momentum-space probability amplitude $\phi(p)$ that corresponds to the position-space probability amplitude

$$\psi(x) = N\mathrm{e}^{-\lambda|x|}.$$

Normalize the wave function first. You may use results from a previous problem set.

(b) Calculate the uncertainty in momentum Δp. Use your result from a previous problem to compute the product $\Delta x \Delta p$. You may use results from a previous problem set. Does it saturate the Heisenberg uncertainty principle?

$\chi(x)$

Hermitian conjugates of operators.
Recall from class that by definition the hermitian conjugate $\hat{A}^\dagger$ of an operator $\hat{A}$ satisfies

$$\langle\psi|\hat{A}^\dagger|\phi\rangle \equiv \langle\phi|\hat{A}|\psi\rangle^*$$

for any $|\psi\rangle$ and $|\phi\rangle$. In position space, this becomes

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \psi^*(x)(\hat{A}^\dagger)_{\mathrm{pos}}\phi(x) \equiv \left[\int_{-\infty}^{\infty} \mathrm{d}x\, \phi^*(x)(\hat{A})_{\mathrm{pos}}\psi(x)\right]^* .$$

where you may assume that $\psi(x)$ and $\phi(x)$ go to zero faster than $x^{-1/2}$ at $\pm\infty$.

(a) Using the position-space representation, show that the position operator $\hat{X}$ is hermitian, i.e.,, that $\hat{X}^\dagger = \hat{X}$.

(b) Consider an operator $\hat{D}$ whose position-space representation is the derivative operator $(\hat{D})_{\mathrm{pos}} = \frac{\mathrm{d}}{\mathrm{d}x}$. What is the position-space representation of $\hat{D}^\dagger$?

(c) Use your result in part (b) to show that the momentum operator in position-space is a hermitian operator.

Operators in momentum space.
Since the momentum-space probability amplitude $\phi(p)$ contains the same information about the quantum-mechanical state that $\psi(x)$ does, one may calculate all expectation values using $\phi(p)$ rather than $\psi(x)$,

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \;\; \phi^*(p)(\hat{A})_{\mathrm{mom}}\phi(p)$$

where $(\hat{A})_{\mathrm{mom}}$ is the operator corresponding to the observable A in the momentum-space representation.

(a) Compute $(\hat{X})_{\mathrm{mom}}$.

(b) Calculate the commutator $[\hat{X}, \hat{P}]$ using the momentum-space representation of the operators. Does it agree with the commutator calculated using the position-space representation?