Let's determine form of momentum eigenstate $|p\rangle$ in position space, i.e.

$$\psi_p(x) = \langle x|p\rangle$$

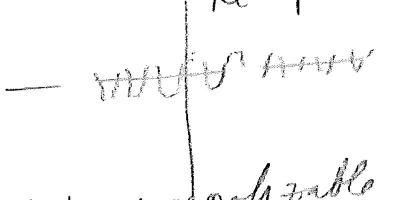
$$\hat{p}\,|p\rangle = p\,|p\rangle$$

$$\underbrace{\langle x|\hat{p}\,|p\rangle}_{p_{pos}\langle x|p\rangle} \quad p\,\langle x|p\rangle$$

$$\frac{\hbar}{i}\frac{d}{dx}\psi_p = p\,\psi_p$$

$$\psi_p = A\,e^{\frac{ipx}{\hbar}}$$

$$\mathrm{Re}\,\psi_p = A\cos\left(\frac{px}{\hbar}\right)$$



Not normalizable in the ordinary sense

$$\int_{-\infty}^{\infty} dx\,|\psi_p|^2 = \int_{-\infty}^{\infty} dx\,|A|^2 = \infty$$

Prob. density $|\psi_p(x)|^2 = |A|^2$ same everywhere

equally likely to be anywhere!

$|p\rangle$ form an orthonormal basis in the Dirac sense

ie; $\langle p'|p\rangle = \delta(p-p')$
Use this to determine $A$.

$$\delta(p-p') = \int_{-\infty}^{\infty} dx \, \langle p'|x\rangle\langle x|p\rangle$$

$$= \int_{-\infty}^{\infty} dx \, A^* e^{-\frac{ip'x}{\hbar}} \, A e^{\frac{ipx}{\hbar}}$$

$$= |A|^2 \int_{-\infty}^{\infty} dx \, e^{\frac{i(p-p')x}{\hbar}}$$

[Can "see" that r.h.s $= 0$ if $p \neq p'$ because cos + i sin = area vanishes.
But need to choose particular $|A|^2$ for it to be normalized]

$$= |A|^2 \lim_{L\to\infty} \underbrace{\int_{-L}^{L} dx \, e^{\frac{i(p-p')x}{\hbar}}}$$

$$\frac{\hbar}{i(p-p')} \underbrace{e^{\frac{i(p-p')x}{\hbar}}\Big|_{-L}^{L}}_{2i\sin\frac{(p-p')L}{\hbar}}$$

$$\delta(p-p') = 2\hbar |A|^2 \lim_{L\to\infty} \frac{\sin\frac{(p-p')L}{\hbar}}{(p-p')}$$

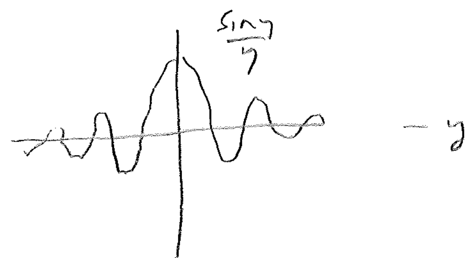
$$\delta(p-p') = 2\hbar |A|^2 \lim_{L\to\infty} \frac{\sin\left(\frac{(p-p')L}{\hbar}\right)}{p-p'}$$

Integrate both sides wrt $p$

$$\underbrace{\int_{-\infty}^{\infty}\delta(p-p')\,dp}_{1} = 2\hbar|A|^2 \lim_{L\to\infty} \underbrace{\int_{-\infty}^{\infty} \frac{dp}{p-p'}\sin\left(\frac{(p-p')L}{\hbar}\right)}$$

Let $y = \frac{(p-p')L}{\hbar}$

$$\int_{y=-\infty}^{\infty}\frac{dy}{y}\sin y$$




Contour integration (or numerical) gives $\pi$

$$1 = 2\pi\hbar|A|^2$$

$$A = \frac{1}{\sqrt{2\pi\hbar}}$$

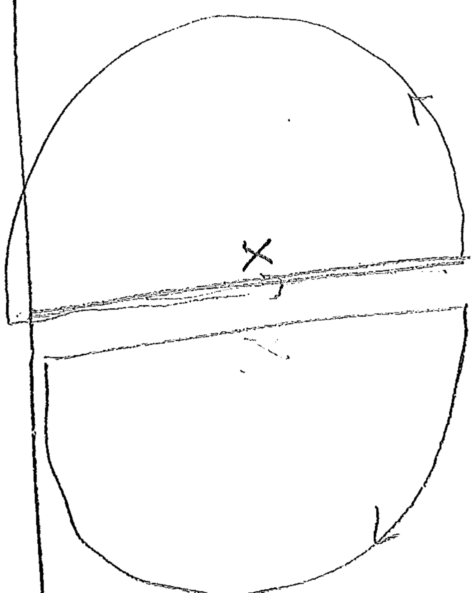
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}$$

$$\left[\ U|x\rangle = e^{-\frac{ia\hat{p}}{\hbar}}\frac{1}{\sqrt{2\pi\hbar}}\int dp\, e^{-\frac{ipx}{\hbar}}|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\, e^{-ip\frac{(x+a)}{\hbar}}|p\rangle = |x+a\rangle\ \right]$$

(assume $\epsilon > 0$)

$$\int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{\sin y}{y - i\epsilon}$$

$$= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{2i}\left(\frac{e^{iy} - e^{-iy}}{y - i\epsilon}\right)$$



$$= \lim_{\epsilon \to 0} \frac{1}{2i} \underbrace{\oint_{UHP} \frac{e^{iy}}{y - i\epsilon}} - \frac{1}{2i} \underbrace{\oint_{LHP} \frac{e^{-iy}}{y - i\epsilon}}_{0}$$

Pole is at $y = i\epsilon$

$2\pi i \,\text{Res}$

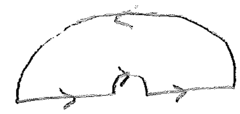
$= 2\pi i\, e^{-\epsilon}$

$= \pi$ ✓

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_{-\infty}^{-\epsilon} \frac{\sin x}{x}\,dx + 2\epsilon + \int_{\epsilon}^{\infty} \frac{\sin x}{x}\,dx \qquad \text{since } \frac{\sin x}{x} \to 1$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

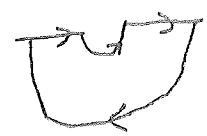
$$I = I_1 + I_2 + 2\epsilon \quad \text{where} \begin{cases} I_1 = \frac{1}{2i}\int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \frac{1}{2i}\int_{\epsilon}^{\infty} \frac{e^{ix}}{x}\,dx \\ I_2 = -\frac{1}{2i}\int_{-\infty}^{-\epsilon} \frac{e^{-ix}}{x}\,dx - \frac{1}{2i}\int_{\epsilon}^{\infty} \frac{e^{-ix}}{x}\,dx \end{cases}$$

Consider $J_1 = \frac{1}{2i}\oint \frac{e^{iz}}{z}\,dz$ on contour 

$z = Re^{i\theta}$
$dz = Re^{i\theta}\,i\,d\theta$  $\frac{1}{2i}\int \frac{e^{iz}}{z}\,dz = \frac{1}{2i}\int i\,d\theta\, e^{iRe^{i\theta}} = \int \frac{d\theta}{2}\, e^{iR(\cos\theta + i\sin\theta)} \xrightarrow[R\to\infty]{} 0$ because $\sin\theta > 0$

$\frown \; = \frac{1}{2i}\int i\,d\theta\, e^{i\epsilon e^{i\theta}} = \int_{\pi}^{0} \frac{d\theta}{2} = -\frac{\pi}{2}$

$\Rightarrow J_1 = I_1 + 0 - \frac{\pi}{2}$ but $J_1 = 0$ because no poles enclosed $\Rightarrow I_1 = \frac{\pi}{2}$

Consider $J_2 = -\frac{1}{2i}\oint \frac{e^{-iz}}{z}\,dz$ on contour 

 $z = Re^{i\theta}$
$dz = Re^{i\theta}\,i\,d\theta$ $\quad -\int \frac{d\theta}{2}\, e^{-iR(\cos\theta + i\sin\theta)} \xrightarrow[R\to\infty]{} 0$ because $\sin\theta < 0$

$\smile \; -\int_{\pi}^{2\pi} \frac{d\theta}{2}\, e^{i\epsilon e^{i\theta}} = -\frac{\pi}{2}$

$J_2 = I_2 + 0 - \frac{\pi}{2}$ but $J_2 = 0$ so $I_2 = \frac{\pi}{2}$

$I = I_1 + I_2 + 2\epsilon \longrightarrow \pi.$

From $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$ we can get $P_{pos}$ & $X_{mom}$

$$P_{pos}\langle x|\psi\rangle = \langle x|\hat{P}|\psi\rangle$$
$$= \int dp\, \langle x|p\rangle\langle p|\hat{P}|\psi\rangle$$
$$= \frac{1}{\sqrt{2\pi\hbar}}\int dp\, \underbrace{p\, e^{\frac{ipx}{\hbar}}}_{\frac{\hbar}{i}\frac{d}{dx} e^{\frac{ipx}{\hbar}}} \langle p|\psi\rangle$$
$$= \frac{\hbar}{i}\frac{d}{dx}\frac{1}{\sqrt{2\pi\hbar}}\int dp\, e^{\frac{ipx}{\hbar}}\langle p|\psi\rangle$$
$$= \frac{\hbar}{i}\frac{d}{dx}\int dp\, \langle x|p\rangle\langle p|\psi\rangle$$
$$= \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle$$

Also

$$X_{mom}\langle p|\psi\rangle = \langle p|X|\psi\rangle$$
$$= \int dx\, \langle p|x\rangle\langle x|\hat{X}|\psi\rangle$$
$$= \frac{1}{\sqrt{2\pi\hbar}}\int dx\, \underbrace{x\, e^{-\frac{ipx}{\hbar}}}_{i\hbar\frac{d}{dp} e^{-\frac{ipx}{\hbar}}} \langle x|\psi\rangle$$
$$= i\hbar\frac{d}{dp}\int dx\, \langle p|x\rangle\langle x|\psi\rangle$$
$$= i\hbar\frac{d}{dp}\langle p|\psi\rangle$$