Momentum space

Momentum is an observable

associated w/ a hermitian operator $\hat{p}$

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

p = possible results of measurement of momentum, $p \in \mathbb{R}$

$|p\rangle$ = state of perfectly well defined momentum (idealization)

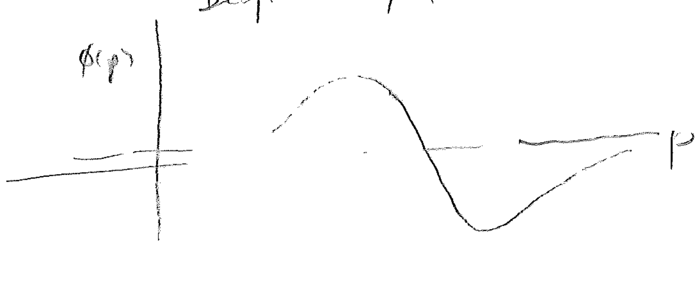
Completeness $\int dp\, |p\rangle\langle p| = 1$

Orthonormality $\langle p'|p\rangle = \delta(p-p')$

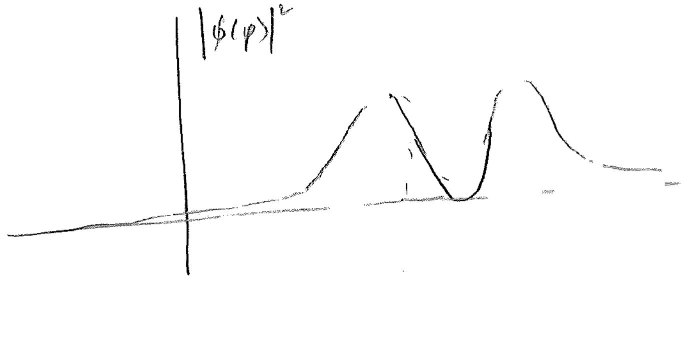
Arbitrary state

$$|\psi\rangle = \int dp\; |p\rangle\langle p|\psi\rangle$$

Define $\langle p|\psi\rangle = \phi(p)$ = momentum space wavefunction (can be complex)



$|\phi(p)|^2$ = probability density in momentum space



$|\phi(p)|^2 dp$ = prob of measuring momentum between p and $p+dp$

[Consider using $\tilde{\psi}(p)$ instead of $\phi(p)$]

$$|\psi\rangle = \int dp\, |p\rangle \phi(p) \implies \langle\psi| = \int dp\, \phi^*(p) \langle p|$$

Normalization

$$1 = \langle\psi|\psi\rangle = \left(\int dp\, \phi^*(p) \langle p|\right)\left(\int dp'\, |p'\rangle \phi(p')\right)$$

$$= \int dp\, \phi^*(p) \underbrace{\int dp' \underbrace{\langle p|p'\rangle}_{\delta(p'-p)} \phi(p')}_{\phi(p) \text{ by def of } \delta.}$$

$$= \int dp\, |\phi(p)|^2 = 1$$

but quicker to only expand in $|\psi\rangle$

$$\langle\psi|\psi\rangle = \langle\psi| \int dp\, |p\rangle \phi(p)$$

$$= \int dp\, \langle\psi|p\rangle \phi(p)$$

$$= \int dp\, |\phi(p)|^2 = 1$$

$\phi(p)$ is square integrable (in momentum space)

expectation values

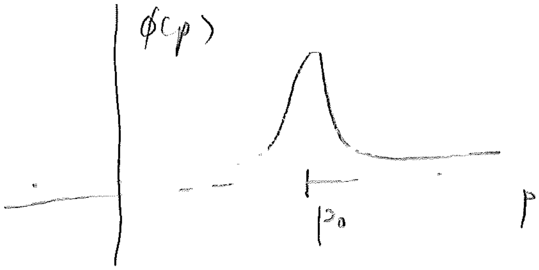
$$\langle p\rangle = \langle\psi|\hat{P}|\psi\rangle = \langle\psi| \int dp\, \underbrace{\hat{P}|p\rangle}_{p|p\rangle} \phi(p)$$

$$= \int dp\, p \langle\psi|p\rangle \phi(p)$$

$$= \int dp\, p\, |\phi(p)|^2$$

abs.	mom space
$\|\psi\rangle$	$\phi(p)$
$\langle\psi\|$	$\phi^*(p)$
$\hat{P}$	$P_{mom} = p$

Since states of perfectly well defined momenta are physically unrealizable, consider a state of a distribution of momenta about some avg momentum p_0



We like Gaussian distribution because they are easy to integrate!

$$\phi(p) = N e^{-\beta(p-p_0)^2}$$

N determined by normalizing

$$1 = \int dp\, |\phi(p)|^2 = N^2 \underbrace{\int_{-\infty}^{\infty} dp\, e^{-2\beta(p-p_0)^2}}_{\underbrace{\int_{-\infty}^{\infty} dq\, e^{-2\beta q^2}}_{\sqrt{\frac{\pi}{2\beta}}}}$$

Let $q = p - p_0$
$dq = dp$

$\sqrt{\frac{\pi}{2\beta}}$ as well see in a moment

$$\Rightarrow N = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}}$$

$$\Rightarrow \boxed{\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}}$$ normalized gaussian

Doing Gaussian integrals

$$I(a) = \int_{-\infty}^{\infty} dx\, e^{-ax^2}$$

Can't use $u = x^2$ because $du = 2x\,dx$

Consider

$$I(a)^2 = \int_{-\infty}^{\infty} dx\, e^{-ax^2} \int_{-\infty}^{\infty} dy\, e^{-ay^2}$$

$$= \iint dx\,dy\, e^{-a(x^2+y^2)}$$

Switch to polar coords $x = r\cos\phi$
$y = r\sin\phi$
$dx\,dy = r\,dr\,d\phi$

$$= \int_0^{\infty} r\,dr \int_0^{2\pi} d\phi\, e^{-ar^2}$$

$$= \pi \int_0^{\infty} 2r\,dr\, e^{-ar^2}$$

$u = r^2 \Rightarrow du = 2r\,dr$

$$I(a)^2 = \pi \int_0^{\infty} du\, e^{-au} = -\frac{\pi}{a} e^{-au}\Big|_0^{\infty} = \frac{\pi}{a}$$

$$\boxed{I(a) = \sqrt{\frac{\pi}{a}}}$$

Consider more general integral for later

$$I(a,b) = \int_{-\infty}^{\infty} dx \; e^{-ax^2 - bx}$$

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

$$= a\left(\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a^2}\right)$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

$$I(a,b) = \int_{-\infty}^{\infty} dx \; e^{-a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}}$$

$$= e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} dy \; e^{-ay^2} \qquad \left(y = x + \frac{b}{2a}\right)$$

$$= e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$$

$$I(a,b,c) = \int_{-\infty}^{\infty} dx \; e^{-ax^2 - bx - c}$$

$$= e^{\left(\frac{b^2}{4a} - c\right)} \sqrt{\frac{\pi}{a}}$$

$$I(a) = \int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}} = \sqrt{\pi}\, a^{-\frac{1}{2}}$$

Differentiate wrt. parameter a

$$\frac{dI}{da} = \int_{-\infty}^{\infty} dx \, (-x^2) e^{-ax^2} = -\frac{1}{2}\sqrt{\pi}\, a^{-3/2}$$

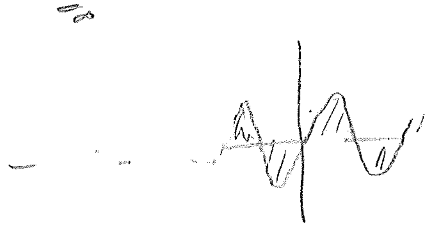
$$\Rightarrow \int_{-\infty}^{\infty} dx \, x^2 e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Can repeat to get all even powers [HW]

$$\int_{-\infty}^{\infty} dx \, x^{2n} e^{-ax^2}$$

All odd powers give vanishing integral

$$\int_{-\infty}^{\infty} dx \, x^{2n+1} e^{-ax^2} = 0$$



Return to gaussian momentum distribution

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$

Expectation value

$$\langle p \rangle = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp \; p \; e^{-2\beta(p-p_0)^2} \qquad q = p - p_0$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dq \; (p_0 + q) \, e^{-2\beta q^2}$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} p_0 \underbrace{\int_{-\infty}^{\infty} dq \; e^{-2\beta q^2}}_{\left(\frac{\pi}{2\beta}\right)^{\frac{1}{2}}} + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} dq \; q e^{-4\beta q^2}}_{0}$$

$$= p_0$$

$$\langle p^2 \rangle = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int dp \; p^2 e^{-2\beta(p-p_0)^2}$$

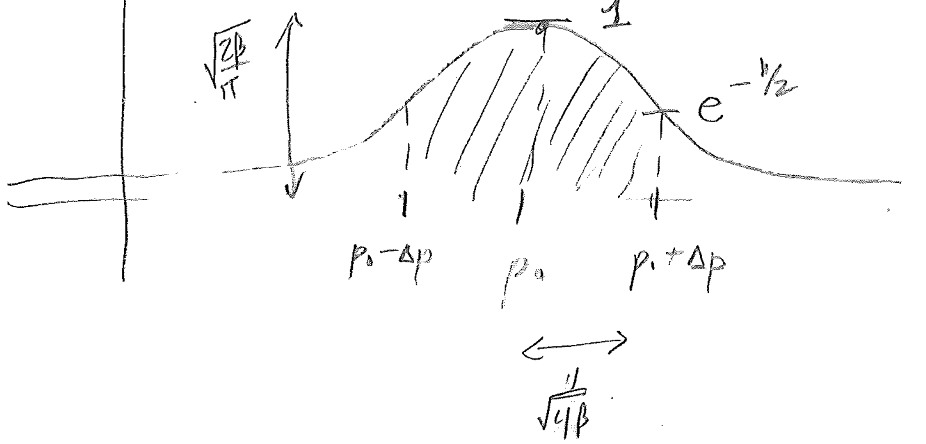
$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int dq \; (p_0^2 + 2p_0 q + q^2) \, e^{-2\beta q^2}$$

$$= p_0^2 + 0 + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int dq \; q^2 e^{-2\beta q^2}}_{\frac{1}{2}\left(\frac{\pi}{(2\beta)^3}\right)^{\frac{1}{2}}}$$

$$= p_0^2 + \frac{1}{4\beta}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{1}{\sqrt{4\beta}}$$

$$|\phi(p)|^2 = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} e^{-2\beta(p-p_0)^2}$$



as $\beta \to \infty$
gets narrower & higher

$$e^{-2\beta(\Delta p)^2} = e^{-\frac{1}{2}} \approx 0.6065$$

prob of momentum being between $p_0 - \Delta p$ and $p_0 + \Delta p$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{p_0-\Delta p}^{p_0+\Delta p} dp \; e^{-2\beta(p-p_0)^2}$$

$$= 2\left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_0^{\frac{1}{\sqrt{4\beta}}} dq \; e^{-2\beta q^2}$$

Let $t = \sqrt{2\beta}\, q$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} dt \; e^{-t^2}$$

$$= \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) = .683$$

Expectation value and uncertainty in position. Consider the wavefunction

$$\psi(x) = A e^{-\lambda |x - x_0|},$$

where λ is real and positive.

(a) Normalize the wavefunction, and sketch the probability density.

(b) Compute the expectation value of position.

(c) Compute the uncertainty Δx in position.

(d) Calculate the probability that the particle is found in the region $|x - x_0| < \Delta x$. How does this compare with the result for a Gaussian distribution found in class?

Gaussian in momentum space.
Calculate the expectation value of p^{2n} (where n is an arbitrary positive integer) for the Gaussian wavefunction

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{1/4} \mathrm{e}^{-\beta(p-p_0)^2}$$

using the method discussed in class (differentiation with respect to a parameter). [Hint: you may find it useful to know the definition of the double factorial: $N!! = N(N-2)(N-4)(N-6)\cdots$, so for example $6!! = 48$.]