<u>Momentum</u>: generator of spatial translations

Recall: rotation about axis $\hat{n}$ through angle ϕ generated by

$$U = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\phi}$$ where $\vec{J}$: angular mom. operator

Time evolution (translation in time by t) generated by

$$U = e^{-\frac{i}{\hbar}Ht}$$ where H = hamiltonian (energy) operator

Spatial translations by $\vec{a}$ generated by

$$U = e^{-\frac{i}{\hbar}\vec{P}\cdot\vec{a}}$$ where $\vec{P}$: momentum operator

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These operators are all unitary (ie $U^{\dagger}U = 1$)
generators are hermitian operators

eg $U = e^{-\frac{i}{\hbar}\vec{P}\cdot\vec{a}}$

$$U^{\dagger} = \left(e^{-\frac{i}{\hbar}\vec{P}\cdot\vec{a}}\right)^{\dagger} = e^{\left(-\frac{i}{\hbar}\vec{P}\cdot\vec{a}\right)^{\dagger}} = e^{\frac{i}{\hbar}\vec{P}\cdot\vec{a}}$$

$$= e^{-\frac{i}{\hbar}\vec{P}\cdot(-\vec{a})} = U^{-1}$$

so $U^{\dagger}U = U^{-1}U = 1$

Restrict to one dimension for now

$U = e^{-\frac{iaP}{\hbar}}$ translates state in $+x$ direction by a

$$\boxed{U|x\rangle = |x+a\rangle}$$

How does U act on $|\psi\rangle$?

Consider $U^\dagger|x\rangle = U^{-1}|x\rangle = |x-a\rangle$

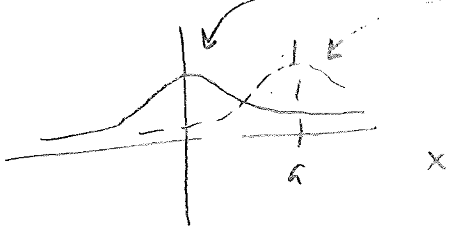
Then $\langle\psi|U^\dagger|x\rangle = \langle\psi|x-a\rangle$

Take h.c. $\underbrace{\langle x|U|\psi\rangle}_{U_{pos}\langle x|\psi\rangle} = \langle x-a|\psi\rangle = \psi(x-a)$

Thus $\boxed{U_{pos}\,\psi(x) = \psi(x-a)}$

[Minus sign seems strange but consider

$\psi(x) = e^{-\alpha x^2}$ $\qquad \psi(x-a) = e^{-\alpha(x-a)^2}$



Indeed wavepacket has been translated to the right]

[Alternative proof: $U|\psi\rangle = \int dx\, U|x\rangle\,\psi(x)$

$= \int dx\, |x+a\rangle\,\psi(x)$ $\quad y = x+a$

$= \int dy\, |y\rangle\,\psi(y-a)$ $\quad x = y$

$= \int dx\, |x\rangle\,\psi(x-a)$]

$\langle x|U|\psi\rangle = \psi(x-a)$

Observe

$$\psi(x-a) = \psi(x) - a\frac{d\psi}{dx} + \frac{1}{2}a^2\frac{d^2\psi}{dx^2} - \frac{1}{3!}a^3\frac{d^3\psi}{dx^3} + \cdots$$

$$= e^{-a\frac{d}{dx}}\psi(x)$$

Then

$$U_{pos}\,\psi(x) = \psi(x-a)$$

$$e^{-\frac{ia}{\hbar}P_{pos}}\psi(x) = e^{-a\frac{d}{dx}}\psi(x)$$

$$P_{pos} = \frac{\hbar}{i}\frac{d}{dx}$$

That is

$$\langle x|P|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle$$

$$= \frac{\hbar}{i}\frac{d\psi}{dx}$$

Consider $[X, P]_{pos}\, \psi(x) = x\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi - \frac{\hbar}{i}\underbrace{\frac{d}{dx}(x\psi)}_{\psi + x\frac{d\psi}{dx}}$

$$= i\hbar\,\psi$$

Since this is valid for arbitrary ψ,

$$[x, p]_{pos} = i\hbar$$

In general

$$[\hat{X}, \hat{P}] = i\hbar\,\mathbb{1} \qquad \text{Heisenberg commutation relation}$$

$\hat{X}, \hat{P}$ are incompatible

Recall generalized uncertainty principle

$$\Delta A\ \Delta B \geq \frac{1}{2}\left|\langle\psi|\,[A,B]\,|\psi\rangle\right|$$

$$\Delta x\ \Delta p \geq \frac{1}{2}\underbrace{\left|\langle\psi|\,[x,p]\,|\psi\rangle\right|}_{|i\hbar\langle\psi|\psi\rangle|}$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

Prob: $[x, p^n]$
$[x^n, p]$
$[V(x), p] = i\hbar\, V'(x)$

Commutators.
(a) Compute $[X, P^n]$ for arbitrary positive integer n.
(Hint: first show that $[A, B^n] = \sum_{m=0}^{n-1} B^m [A, B] B^{n-m-1}$, a generalization of an identity you proved earlier.)
(b) Compute $[X^n, P]$ for arbitrary positive integer n.
(c) Show that $[V(X), P] = i\hbar(dV/dX)$.
(Hint: use the Taylor expansion of $V(x)$ together with result in part (b).)

Operator approach to the harmonic oscillator. There is a very elegant way to solve the harmonic oscillator using the operator approach (introduced, of course, by Dirac), related to the approach we have already used for spin angular momentum. This problem will guide you through this approach. The Hamiltonian is

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2.$$

where ω is the natural angular frequency of the harmonic oscillator.

(a) Begin by defining the operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{i}{m\omega}P\right).$$

Note that a is *not* hermitian. Using $[X, P] = i\hbar$, calculate the commutator $[a, a^\dagger]$.

(b) Define $N = a^\dagger a$. Show that $N = (H/\hbar\omega) - \frac{1}{2}$.

(c) Calculate $[N, a]$ and $[N, a^\dagger]$. (Use the commutator identities you proved in a previous problem set.)

(d) Let $|\lambda\rangle$ be eigenstates of N with eigenvalues λ. (Since N is hermitian, the eigenvalues are real numbers.) Using the results of (c), show that a and $a^\dagger$ act as lowering and raising operators respectively on $|\lambda\rangle$.

(e) Show that $\lambda \geq 0$. To do so, evaluate the expectation value of N for the state $|\lambda\rangle$, and rewrite this as the norm of another state.

(f) Since λ is bounded below, there must be an eigenstate with minimal value of λ which cannot be lowered: $a|\lambda_{\text{min}}\rangle = 0$. (If the right hand side were not zero, it would be a state with eigenvalue $\lambda_{\text{min}} - 1$, which is impossible.) Show that this equation implies that $\lambda_{\text{min}} = 0$.

(g) The other possible values of λ are positive integers, obtained by acting on $|\lambda_{\text{min}}\rangle$ with raising operators. (Convince yourself that non-integer values of λ are unacceptable, by acting repeatedly on such states with lowering operators.) Therefore, we relabel the eigenstates as $|n\rangle$ where $N|n\rangle = n|n\rangle$ for any non-negative integer n. Using the result of part (b), what is the energy of the state $|n\rangle$?

(h) Define $a|n\rangle = c(n)|n-1\rangle$ and $a^\dagger|n\rangle = d(n)|n+1\rangle$. You may assume that the eigenstates are normalized. Compute $c(n)$ and $d(n)$. (You may take them to be positive real numbers.)

(i) The eigenstate $|n\rangle$ can be expressed by acting repeatedly on the ground state with the raising operator

$$|n\rangle = C(a^\dagger)^n|0\rangle.$$

Compute the value of the constant C.