[Position observable]

Review [on side board]

Previously

observable A $\rightarrow$ hermitian operator $\hat{A}$

For observables w/ discrete spectra — finite or infinite

eigenvalue eqn $\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$

eigenvalues : results of possible measurements of A

eigenstates : state of determinate A

orthonormality $\langle a_n | a_m \rangle = \delta_{nm}$

completeness $\sum_n |a_n\rangle\langle a_n| = \mathbb{1}$

decomposition of an arbitrary state into eigenstates

$$|\psi\rangle = \sum |a_n\rangle\langle a_n|\psi\rangle = \sum \psi_n |a_n\rangle$$

$$\psi_n = \langle a_n|\psi\rangle = \text{prob. amplitude}$$

How does $\hat{A}$ act on $|\psi\rangle$?

$$\hat{A}|\psi\rangle = \sum \hat{A}\,|a_n\rangle\langle a_n|\psi\rangle = \sum a_n\psi_n\,|a_n\rangle$$

$|\psi\rangle$ is normalized

$$\Rightarrow 1 = \langle\psi|\psi\rangle = \sum \langle\psi|a_n\rangle\langle a_n|\psi\rangle = \sum \psi_n^*\psi_n = |\psi_n|^2$$

Expectation value

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \sum a_n\psi_n \underbrace{\langle\psi|a_n\rangle}_{\psi_n^*} = \sum a_n|\psi_n|^2$$

$$\Rightarrow |\psi_n|^2 = \text{prob. of measuring } a_n$$

Position observable $\underline{x}$ (one dimension for now)

Position operator $\hat{X}$

Eigenvalue eqn $\hat{X}|x\rangle = x|x\rangle$

$x \in \mathbb{R}$ denotes possible results of measurement of position

$|x\rangle$ = state w/ perfectly well defined position

(unphysical idealization, not normalizable as we'll see)

Completeness relation

recall, for a discrete spectrum $\{a_n\}$ $\quad \sum_n |a_n\rangle\langle a_n| = 1$

for a continuous spectrum

$$\int dx \; |x\rangle\langle x| = 1$$

Orthonormality:

recall, for a discrete spectrum $\langle a_n|a_m\rangle = \delta_{nm}$ ← Kronecker

for a continuous spectrum

$$\langle x|x'\rangle = \delta(x-x')$$

↑ Dirac delta function

(we'll define this later)

Consider a normalized state $|\psi\rangle$

$$1 = \langle\psi|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \int dx\, \underbrace{\langle\psi|x\rangle}_{\langle x|\psi\rangle^*}\langle x|\psi\rangle = \int dx\, |\langle x|\psi\rangle|^2$$

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \int dx\, \langle\psi|\hat{x}|x\rangle\langle x|\psi\rangle = \int dx\, \langle\psi|x|x\rangle\langle x|\psi\rangle = \int dx\, x\,|\langle x|\psi\rangle|^2$$

Recall from p21.40 that a normalized wave function obeys

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1 \quad \text{and} \quad \langle x\rangle = \int_{-\infty}^{\infty} dx\, x\,|\psi(x)|^2$$

↳ implies $\psi(x) \xrightarrow[x\to\pm\infty]{} 0$ faster than $\frac{1}{\sqrt{x}}$ (square integrable)

This suggests $\langle x|\psi\rangle = \psi(x)$ = probability amplitude that particle has position x

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$$|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi(x)\, |x\rangle$$

ie $\psi(x)$ = coefficient of $|x\rangle$ in the x-basis

$$\langle\psi| = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \langle x|$$

Define an operator $\hat{A}$ in position space by

$$\langle x|\hat{A}|\psi\rangle \equiv A_{pos}\langle x|\psi\rangle$$

Thus $\underbrace{\langle x|\hat{x}}_{\langle x|x}|\psi\rangle = X_{pos}\langle x|\psi\rangle$

$$\Rightarrow \quad X_{pos} = x \quad \text{(multiply by } x\text{)}$$

abstract	position space
$\vert\psi\rangle$	$\psi(x)$
$\langle\psi\vert$	$\psi^*(x)$
$\hat{X}$	x
$\langle\psi\vert\,\hat{x}\,\vert\psi\rangle$	$\int_{-\infty}^{\infty} dx\ \psi^*(x)\, x\, \psi(x)$

bracket $\Rightarrow$ integration (for spin, $\Rightarrow$ matrix mult.)

[For self: $\hat{X} = \int dx\ \hat{X}\, |x\rangle\langle x| = \int dx\ |x\rangle\, x\, \langle x|$]

What about orthonormality of eigenkets $|x\rangle$?

If $x \neq x'$, expect $\langle x' | x \rangle = 0$

If $x = x'$, expect $\langle x | x \rangle = 1$??

This won't work. Consider arbitrary state

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \; |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx \; |x\rangle \, \psi(x)$$

Then

$$\psi(x_0) = \langle x_0 | \psi \rangle = \int_{-\infty}^{\infty} dx \; \langle x_0 | x \rangle \, \psi(x)$$

If $\langle x | x \rangle = 1$ this integral vanishes

Need $\langle x | x \rangle$ much "bigger"!

Write $\langle x_0 | x \rangle = \delta(x - x_0)$ = Dirac delta "function"
(actually a distribution)

We must have

$$\begin{cases} ① \quad \delta(x - x_0) = 0 \quad \text{if} \quad x \neq x_0 \\ ② \quad \psi(x_0) = \int_{-\infty}^{\infty} dx \; \delta(x - x_0)\, \psi(x) \quad \text{for any } \psi(x) \end{cases}$$

These properties necessary & sufficient to define $\delta(\cdot - x_0)$

Actually can replace ② by

$$③ \quad 1 = \int_{-\infty}^{\infty} dx \; \delta(x - x_0)$$

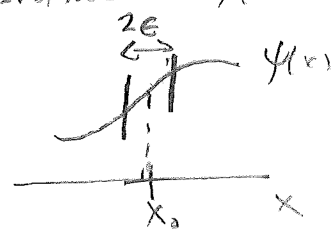
(3) is sufficient to prove (2)

Consider $\int_{-\infty}^{\infty} dx \; \delta(x-x_0)\, \psi(x)$

$$= \int_{x_0-\epsilon}^{x_0+\epsilon} dx \; \delta(x-x_0)\, \psi(x)$$

for $\epsilon > 0$
since $\delta(x-x_0) = 0$
outside limits of integration

Consider arbitrarily small ϵ. Then for continuous $\psi(x)$



$$= \int_{x_0-\epsilon}^{x_0-\epsilon} dx \; \delta(x-x_0)\, \psi(x_0)$$

$$= \psi(x_0) \int_{x_0-\epsilon}^{x_0-\epsilon} dx \; \delta(x-x_0)$$

$$= \psi(x_0) \underbrace{\int_{-\infty}^{\infty} dx \; \delta(x-x_0)}_{1 \text{ by } (3)}$$

$$= \psi(x_0)$$

which is (2)

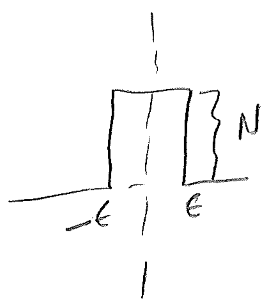
Dirac delta function $\delta(x)$ defined by (setting $x_0 = 0$)

① $\delta(x) = 0$ if $x \neq 0$

② $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$ (equivalent to ① + ② earlier)

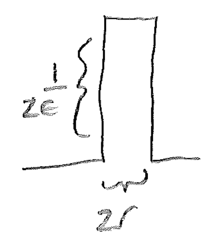
Representation of delta function

Define $R_\epsilon(x) = \begin{cases} 0, & |x| > \epsilon \\ N, & |x| < \epsilon \end{cases}$ "rectangle function"



Require $\int_{-\infty}^{\infty} dx\, R_\epsilon(x) = 1$

This implies $\int_{-\epsilon}^{\epsilon} dx\, N = N(2\epsilon) = 1 \Rightarrow N = \frac{1}{2\epsilon}$



Then we may define $\delta(x) = \lim_{\epsilon \to 0} R_\epsilon(x)$

② $\int_{-\infty}^{\infty} \delta(x)\, dx = \lim_{\epsilon \to 0} \underbrace{\int_{-\infty}^{\infty} R_\epsilon(x)\, dx}_{1} = 1$

① if $x \neq 0$ then $R_\epsilon(x) = 0$ provided $|x| > \epsilon \to 0$

Thus $\delta(0)$ "$=$" ∞

From the representation, we see that $\delta(x)$ is real

and also an even function: $\delta(-x) = \delta(x)$

Since $\delta(x)$ is defined by the properties

$$\begin{cases} \delta(x) = 0 & \text{if } x \neq 0 \\ \int_{-\infty}^{\infty} dx\, \delta(x) = 1 \end{cases}$$

we can formally prove that $\delta(-x) = \delta(x)$ by showing that

$$\begin{cases} \delta(-x) = 0 & \text{if } x \neq 0 \\ \int_{-\infty}^{\infty} dx\, \delta(-x) = 1 \end{cases}$$

First property obvious because if $x \neq 0$ then $-x \neq 0$ so $\delta(-x) = 0$

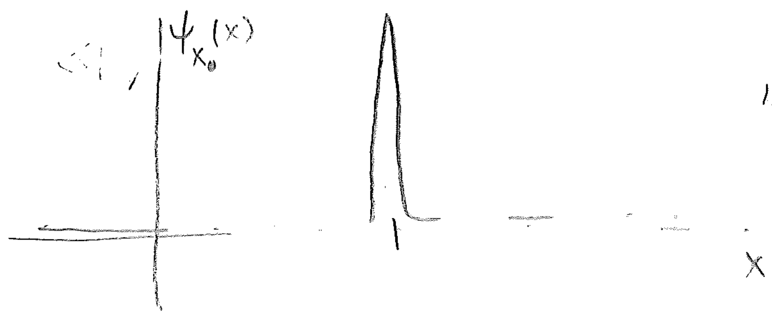
Second property: Let $x = -y$

$$\int_{x=-\infty}^{x=\infty} dx\, \delta(-x) = \int_{y=\infty}^{y=-\infty} (-dy)\, \delta(y) = \int_{y=-\infty}^{y=\infty} dy\, \delta(y) = 1$$

by reality of δ

$$\delta(x-x_0) = \langle x_0 | x \rangle = \langle x | x_0 \rangle$$

Thus $\delta(x-x_0)$ can be (loosely) viewed as the position space wavefunction $\psi_{x_0}(x)$ of the eigenstate $|x_0\rangle$



"perfectly localized"

But $\psi_{x_0}(x)$ is not normalizable!

$$\int_{-\infty}^{\infty} dx \, |\psi_{x_0}(x)|^2 = \int_{-\infty}^{\infty} dx \, \delta(x-x_0)^2$$

$$= \int_{-\infty}^{\infty} dx \, \delta(x-x_0)\, \delta(x-x_0)$$

$$= \delta(0) = \infty \, !$$

Dirac delta function. (FOR ORAL PRESENTATION)
The two defining characteristics of the Dirac delta function are

$$\delta(x) = 0 \text{ if } x \neq 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1.$$

(a) Let c be a real, nonzero constant. Show that

$$\delta(cx) = N\delta(x)$$

and evaluate the constant N. (Note: you should do the $c > 0$ and $c < 0$ cases separately.)
(b) Prove that

$$\lim_{\epsilon \to 0} \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = N\delta(x)$$

where ϵ is a real, nonzero constant. Evaluate the constant N. (Note that it depends on the sign of ϵ.)