(2023)

Matrix representations

Example: S_z-basis for spin $\frac{1}{2}$ particle

Recall $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

$\alpha = \langle +|\psi\rangle$

$\beta = \langle -|\psi\rangle$

We represent $|\psi\rangle$ w/ a "spinor" $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\langle +|\psi\rangle\\ \langle -|\psi\rangle\end{pmatrix}$

Represent $\langle\psi|$ w/ $(\alpha^*, \beta^*) = (\langle\psi|+\rangle, \langle\psi|-\rangle)$

$|+\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$

$|-\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

Represent an operator $\hat{B}$ w/ a matrix $\begin{pmatrix}\langle +|B|+\rangle & \langle +|B|-\rangle\\ \langle -|B|+\rangle & \langle -|B|-\rangle\end{pmatrix}$

More generally, let $\hat{A}$ be a hermitian operator

w/ d eigenstates $|a_n\rangle$, forming an orthonormal basis.

An arbitrary state $|\psi\rangle$ is represented in the A-basis

by a d-dim'l column vector w/ entries $\psi_n = \langle a_n|\psi\rangle$

$$|\psi\rangle \longrightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix} = \begin{pmatrix} \langle a_1|\psi\rangle \\ \vdots \\ \langle a_d|\psi\rangle \end{pmatrix}$$

The corresponding bra $\langle\psi|$ is represented by a row vect w/ entries $\psi_n^* = \langle a_n|\psi\rangle^* = \langle\psi|a_n\rangle$

$$\langle\psi| \longrightarrow (\psi_1^*, \dots, \psi_d^*) = (\langle\psi|a_1\rangle, \dots \langle\psi|a_d\rangle)$$

The eigenstates are simple $|a_n\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \longleftarrow$ nth entry

$$\langle a_n| \longrightarrow (0, 1, 0, 0)$$

Can reconstruct: $|\psi\rangle = \underbrace{\sum |a_n\rangle\langle a_n|}_{\mathbb{1}}\psi\rangle = \sum \psi_n |a_n\rangle$

$$\langle\psi| = \sum \langle\psi|a_n\rangle\langle a_n| = \sum \psi_n^* \langle a_n|$$

Inner product

$$\langle\phi|\psi\rangle = \sum \langle\phi|a_n\rangle\langle a_n|\psi\rangle = \sum \phi_n^* \psi_n$$

$$= (\phi_1^*, \dots, \phi_d^*)\begin{pmatrix}\psi_1\\ \vdots \\ \psi_d\end{pmatrix}$$

Projection operator

$$\hat{\Pi}_n = |a_n\rangle\langle a_n| \longrightarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix}(0\ 1\ 0\ 0) = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

Completeness $\sum \hat{\Pi}_n = \sum |a_n\rangle\langle a_n| = \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} = \mathbb{1}$.

An operator B is represented in the A-basis by a $d \times d$ matrix of matrix elements

$$B_{mn} = \langle a_m | B | a_n \rangle$$

A is represented in the A-basis by a diagonal matrix

$$A_{mn} = \langle a_m | A | a_n \rangle = a_n \langle a_m | a_n \rangle = a_n \delta_{mn}$$

$$A \longrightarrow \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

<u>Product</u> of 2 operators = product of matrices

$$\langle a_m | B C | a_n \rangle = \sum_l \langle a_m | B | a_l \rangle \langle a_l | C | a_n \rangle$$

$$= \sum_l B_{ml} C_{ln}$$

$$= (BC)_{mn}$$

Can reconstruct

$$B = \sum_{m,n} | a_m \rangle B_{mn} \langle a_n |$$

Hermitian conjugate of an operator
corresponds to transpose complex conjugate of matrix

$$|\chi\rangle = B|\psi\rangle \quad \Rightarrow \quad \langle\chi| = \langle\psi| B^\dagger$$

$$\langle\chi|a_n\rangle = \sum_m \langle\psi|a_m\rangle\langle a_m| B^\dagger |a_n\rangle$$

$$\chi_n^* = \sum \psi_m^* (B^\dagger)_{mn}$$

$$\langle a_n|\chi\rangle = \sum_m \langle a_n|B|a_m\rangle\langle a_m|\psi\rangle$$

$$\chi_n = \sum_m B_{nm}\psi_m$$

$$\chi_n^* = \sum B_{nm}^* \psi_m^*$$

$$= \sum \psi_m^* (B^{*T})_{mn}$$

so $\boxed{(B^\dagger)_{mn} = (B^{*T})_{mn}}$

[back to S_z-basis]

$$S_z \longrightarrow \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

Recall $S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$

$$S_x \longrightarrow \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

Recall $S_y = \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right)$

$$S_y = \begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} \frac{3}{4}\hbar^2 & 0 \\ 0 & \frac{3}{4}\hbar^2 \end{pmatrix}$$

Can check $S^2 = S_x^2 + S_y^2 + S_z^2$

Can check $[S_x, S_y] = i\hbar S_z$ etc

Define $S_i = \frac{\hbar}{2}\sigma_i$

σ_i = Pauli spin matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

observe $\sigma_i^\dagger = (\sigma_i^T)^* = \sigma_i \Rightarrow$ Hermitian matrices exercise

(a)
$$[S_x, S_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}\right] = i\hbar S_z.$$

$$[S_y, S_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} & 2i \\ 2i & \end{pmatrix}\right] = i\hbar S_x$$

$$[S_z, S_x] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\hbar S_y$$

(b)
$$\frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right] = \frac{3\hbar^2}{4} I \checkmark$$

Eigenvalue equations

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

In matrix representation

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Any matrix that commutes w/ S_z is also diagonal.

$$[S_z, B] = 0 \quad \Rightarrow \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

$\Rightarrow |\pm\rangle$ are eigenvectors of B.

S_z & B share a complete set of eigenvectors $\Rightarrow$ compatible!
$\Rightarrow$ operators commute

Example: $[S^2, S_z] = 0$

$$S^2 = \begin{pmatrix} \frac{3\hbar^2}{4} & 0 \\ 0 & \frac{3\hbar^2}{4} \end{pmatrix}$$

Eigenvectors of S_x

$$S_x |\psi\rangle = \lambda |\psi\rangle$$

Solve using matrix rep. in S_z basis.

$$S_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$|\psi\rangle \to \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2}\sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left(\frac{\hbar}{2}\sigma_x - \lambda \mathbb{1}\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Trivial solution ($\alpha = \beta = 0$) unless det $= 0$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \implies \lambda = \pm\frac{\hbar}{2}$$

no surprise

$$S_x \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm \frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\sigma_x \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

For $\lambda = +\frac{\hbar}{2}$ we have $\alpha = \beta \Rightarrow |\psi\rangle = \begin{pmatrix}\beta\\ \beta\end{pmatrix}$

Normalize $\langle\psi|\psi\rangle = (\beta^* \; \beta^*)\begin{pmatrix}\beta\\ \beta\end{pmatrix} = 2|\beta|^2 = 1 \Rightarrow \beta = \frac{1}{\sqrt{2}} e^{i\theta}$

overall phase arbitrary : choose $\theta = 0$

$$|S_x = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \leftarrow \text{call this } |+x\rangle$$

Note $|+x\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

For $\lambda = -\frac{\hbar}{2}$, we have $\alpha = -\beta \Rightarrow |\psi\rangle = \begin{pmatrix}-\beta\\ \beta\end{pmatrix}$

no hat!

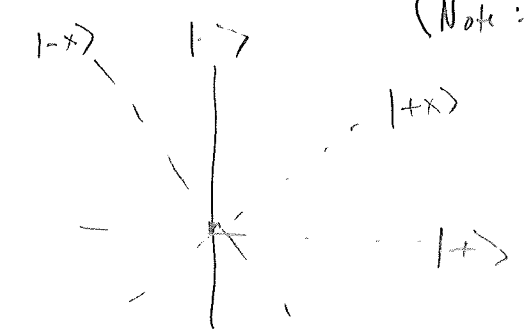
Again choose $\theta = 0$

$$|S_x = -\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \quad \leftarrow \text{call this } |-x\rangle$$

$$|-x\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$|\pm\rangle = S_z$ basis
$|\pm x\rangle = S_x$ basis

(Note: I have a reason for choosing upper component negative: see ahead)



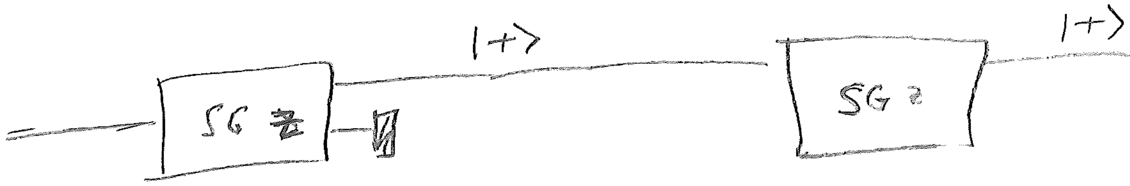
- orthogonal
- 45° rotation in state space [more generally a unitary transformation]

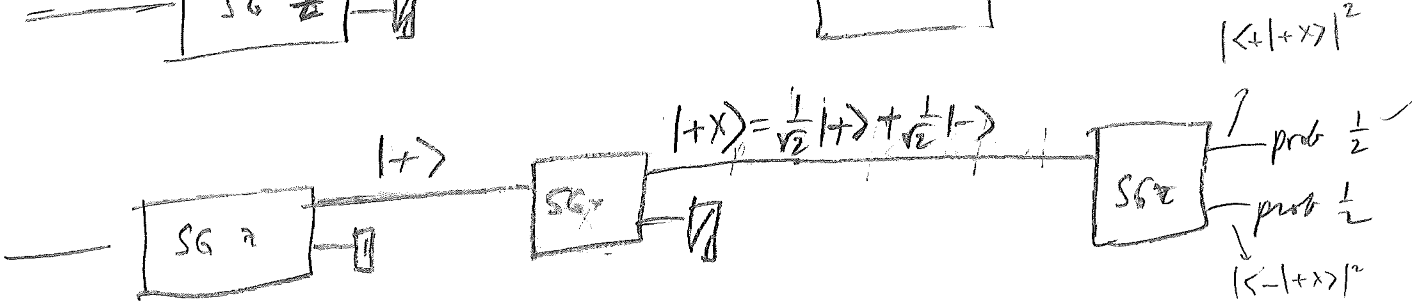
picture only possible because coefficients are (accidentally) real

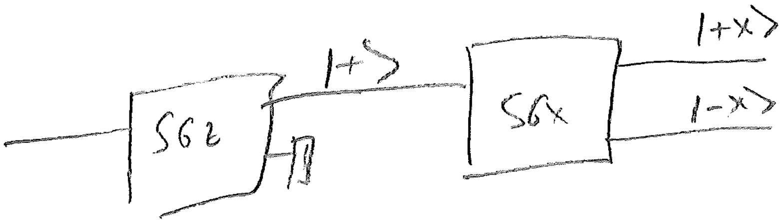
- Because $[S_x, S_z] \neq 0$, S_x + S_z are incompatible and so eigenstates $|\pm x\rangle$ are distinct from $|\pm\rangle \equiv |\pm z\rangle$

- Because $[S^2, S_x] = 0$ (since S^2 prop to identity) S^2 + S_x are compatible and so $|\pm x\rangle$ are also S^2 eigenstates

Return to Stern Gerlach







$\leftarrow$ prob: $|\langle +x|+\rangle|^2 = \frac{1}{2}$

$\leftarrow$ prob $|\langle -x|+\rangle|^2 = \frac{1}{2}$

$$\langle +x|+\rangle = \frac{1}{\sqrt{2}}(1, 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

or

$$\langle +x| = \frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|$$