rev 2023

Compatibility

Two observables A and B are compatible if there exists a complete set of states $|\chi_n\rangle$ that have well-defined values for both A + B. (determinate state)

These states are therefore simultaneous eigenstates of $\hat{A}$ and $\hat{B}$

$$\hat{A}|\chi_n\rangle = a_n|\chi_n\rangle$$
$$\hat{B}|\chi_n\rangle = b_n|\chi_n\rangle$$

Claim: $\hat{A}$ and $\hat{B}$ commute

Proof. Consider an arbitrary state $|\psi\rangle \overset{\text{completeness}}{=} \sum_n c_n|\chi_n\rangle$

Consider $\hat{A}\hat{B}|\psi\rangle = \sum_n c_n \hat{A}\hat{B}|\chi_n\rangle$

$$= \sum c_n \hat{A}\, b_n|\chi_n\rangle$$
$$= \sum c_n b_n \hat{A}|\chi_n\rangle$$
$$= \sum c_n b_n a_n|\chi_n\rangle$$

Similarly $\hat{B}\hat{A}|\psi\rangle = \sum c_n a_n b_n|\chi_n\rangle$

Thus $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$

Since this holds true for an arbitrary state $|\psi\rangle$

we conclude $\hat{A}\hat{B} = \hat{B}\hat{A}$. $\Rightarrow$ $[\hat{A},\hat{B}]=0$

Commutators

[The commutator of $\hat{A}$ and $\hat{B}$ is $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$] already defined earlier 5-1

Compatible observables A + B $\Rightarrow$ $[\hat{A},\hat{B}]=0$

(Converse is also true. A little more work to prove it.)

[HW on commutators]

[After a question by Sean Li]

Proof of converse:

Let $|a_n\rangle$ be a complete set of eigenstates of $\hat{A}$

Consider $\hat{B}|a_n\rangle$.

$\hat{A}\hat{B}|a_n\rangle = \hat{B}\hat{A}|a_n\rangle = a_n \hat{B}|a_n\rangle$

$\therefore \hat{B}|a_n\rangle$ is also an eigenstate of $\hat{A}$ w/ eigenvalue a_n

If $|a_n\rangle$ is a nondegenerate state then $\hat{B}|a_n\rangle \sim |a_n\rangle$

so $|a_n\rangle$ is an eigenstate of $\hat{B}$ as well as $\hat{A}$

If $|a_n\rangle$ is degenerate then $\hat{B}|a_n^{(i)}\rangle = M_{ij}|a_n^{(j)}\rangle$

Diagonalize M_{ij} & you see that the new basis of

$\hat{A}$ eigenstates are also $\hat{B}$ eigenstates

$\Rightarrow \exists$ a complete basis of mutual eigenstates of $\hat{A}$ & $\hat{B}$

$\Rightarrow A$ & B are compatible.

[Contrapositive] If $[\hat{A},\hat{B}] \neq 0$ then A & B are incompatible

If A, B are compatible there exist some states (i.e. the simultaneous eigenstates) for which both $\Delta A = 0$ and $\Delta B = 0$

$\therefore$ $\Delta A \, \Delta B = 0$ for those states

But Heisenberg uncertainty $\Delta x \, \Delta p \geq \frac{\hbar}{2}$ for all states

i.e. X and P are incompatible, and so we can't say $[X,P]=0$

(Later we'll learn $[\hat{X},\hat{P}] = i\hbar \hat{1}$)

In HW, you will prove

Generalized uncertainty relations !

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [\hat{A},\hat{B}] | \psi \rangle \right|$$ [HW]

HW: $[A,B]$ $[B,A]$

$[\alpha A + \beta B, C] = \alpha [A,C] + \beta [B,C]$

$[A, BC] = [A,B]C + B[A,C]$

Jacobi

$[A,B]$ is antihermitian if A, B Hermitian

From now on, drop hats from operators.
[distinction between observable & operator from context]

Recall from Stern Gerlach experiment that
S_z and S_x are incompatible.
(also S_z & S_y and S_y & S_x)

Therefore $[S_x, S_y] \neq 0$

Later, in HW, you'll prove that any form of angular momentum operators satisfies

$$[J_x, J_y] = i\hbar J_z$$

$$[J_y, J_z] = i\hbar J_x$$

$$[J_z, J_x] = i\hbar J_y$$

angular momentum commutation relations
i.e.
commutation relations for the rotation group SO(3)

Comments

- These follow from fact that angular momentum operators are generators of rotations, and rotations don't commute

[
- $\hbar$ makes sense because $\hbar \to 0$ recovers classical physics
- i makes sense because S_i are hermitian but commutator of hermitian matrices is anti-hermitian (you'll prove this in HW)
]

In classical mechanics

continuous symmetries $\Rightarrow$ conserved quantities (observables)

(Noether's theorem)

eg. symmetry under rotations $\Rightarrow$ angular momentum $\vec{J}$ is conserved

(proof uses Lagrangian mechanics)

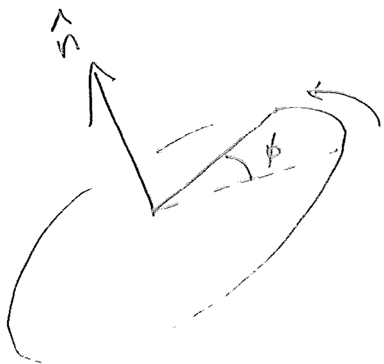
In quantum mechanics, the operators associated

w/ conserved observables "generate" the symmetries.

so $\hat{\vec{J}}$ "generates" rotations, [illegible]

Rotations in 3 dimensions

about axis $\hat{n}$ through angle ϕ $\quad (\hat{n}\cdot\hat{n} = 1)$



[thumb of r.h. along $\vec{n}$; fingers curl]

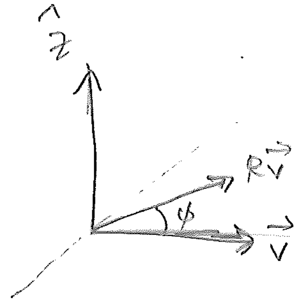
Rotation of a vector $\vec{v}$ is implemented by a special orthogonal 3x3 matrix $R(\hat{n}, \phi)$

orthogonal means $\quad R^T R = \mathbb{1}$
special means $\quad \det R = 1$

Set of rotations forms a <u>Lie group</u> called $SO(3)$

special, orthogonal, 3x3

Consider rotation about z-axis

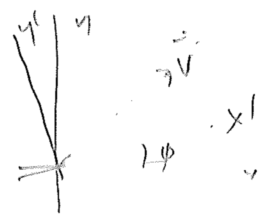


$$R(\hat{z}, \phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

check: $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow R\vec{v} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$ [Find $R(\hat{x},\phi)$ and $R(\hat{y},\phi)$]

"active transformation"

[In P2140, we considered passive transformations
fixed vector; rotated coord sys
so sign is different]



one can express

$$R(\hat{z}, \phi) = e^{-\frac{i\phi}{\hbar} L_z} \mathbb{1}$$

where $L = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ← note: hermitian matrix

$$-\frac{i\phi L}{\hbar} = \begin{pmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{-\frac{i\phi L_z}{\hbar}} = \left[1 + \begin{pmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\phi^2 & 0 & 0 \\ 0 & -\phi^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots \right] \mathbb{1}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ✓$$

We say that L_z "generates" the rotation

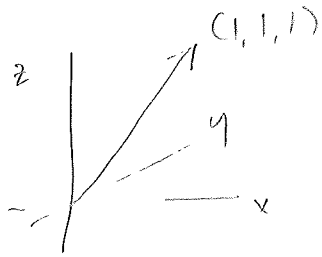
Rotation matrices in 3 dimensions do not commute

[Demonstration w/ 2 sheets of paper]

SO(3) is a nonabelian group

[In HW you will show]

$$[L_x, L_y] = i\hbar L_z$$



[rotation around (1,1,1) take
$x \to y \to z \to x$

$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

(cyclic perms of x, y, z)

We have not explained why L_x, L_y, L_z are angular momenta but Noether's theorem tells us so.

• Later in course, we'll see that $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ & $\vec{p}$ are position & momentum operators obey precisely these relations

∘ Moreover, ~~spin and~~ other forms of angular momenta are also presumed to obey these relations.

$(L_i)_{mn} = i\hbar\,\epsilon_{min}$ $\qquad \epsilon_{123} = 1$

$(L_3)_{12} = i\epsilon_{132} = -i$ ✓

$R(\hat{n},\phi) = e^{-\frac{i\phi}{\hbar}\hat{n}\cdot\vec{L}} = e^{r}$

r : real, anti-symmetric matrix

$r^T = -r$

R : orthogonal $R^T R = 1$

$R(\hat{n},\phi)\,\hat{n} = \hat{n}$ as $r\hat{n} = 0$

$$\underbrace{\begin{pmatrix} 0 & -ic_\theta & is_\theta s_\phi \\ & 0 & -is_\theta c_\phi \\ & & \end{pmatrix}\begin{pmatrix} s\theta c\phi \\ s\theta s\phi \\ c\theta \end{pmatrix}}_{0}$$

$[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k$

$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

3×3 matrices L_i are rotation group

Define "ladder operators"

$J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

Observe $J_\pm$ are not hermitian! $\quad J_\pm^\dagger = J_\mp$

Show $[J_z, J_\pm] = \pm\hbar J_\pm$ [HW]

$$\Rightarrow J_z J_\pm = J_\pm (J_z \pm \hbar)$$

$$\Rightarrow J_z J_\pm^n = J_\pm^n (J_z \pm n\hbar) \qquad \text{[HW]}$$

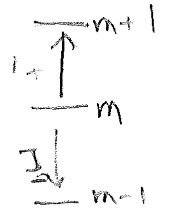
(n any positive integer)

Let $|m\rangle$ be a (normalized) eigenstate of J_z with eigenvalue $m\hbar$

$$J_z J_\pm |m\rangle = J_\pm (m\hbar \pm \hbar)|m\rangle$$

$$= (m \pm 1)\hbar \, J_\pm |m\rangle$$

We have just shown that $J_\pm|m\rangle$ is an eigenstate of J_z w/ eigenvalue $(m\pm1)\hbar$



J_+ is a raising operator: $\quad J_+|m\rangle \sim |m+1\rangle$

J_- is a lowering operator $\quad J_-|m\rangle \sim |m-1\rangle$

$J_\pm|m\rangle$ are not necessarily normalized, so

$$J_+|m\rangle = c_+(m)\,|m+1\rangle$$
$$J_-|m\rangle = c_-(m)\,|m-1\rangle$$

Classically

$$\vec{J}^2 = \vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2$$

QM, direction doesn't make sense because cannot simultaneously define the diff. components of $\vec{J}$

Take our definition of the operator $\hat{J}^2$ to be $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

[HW] Show

$$J_- J_+ = J^2 - J_z^2 - \hbar J_z$$

$$J_+ J_- = J^2 - J_z^2 + \hbar J_z$$

Specialize to spin $\frac{1}{2}$

Consider a spin-$\frac{1}{2}$ particle and let $J \to S$

There are only two eigenstates $|m\rangle$ of S_z:

$m \uparrow$

$$\text{———} \quad |\tfrac{1}{2}\rangle = |+\rangle \qquad\qquad S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

$$\text{———} \quad |-\tfrac{1}{2}\rangle = |-\rangle$$

Apply product to spin $\frac{1}{2}$ eigenstates

$$S_-\underbrace{S_+|+\rangle}_{0} = S^2|+\rangle - \underbrace{S_z^2|+\rangle}_{\frac{\hbar^2}{4}|+\rangle} - \underbrace{\hbar S_z|+\rangle}_{\frac{\hbar^2}{2}|+\rangle}$$

because $|\frac{3}{2}\rangle$ doesn't exist

$$S^2|+\rangle = \frac{3}{4}\hbar^2|+\rangle$$

Similarly $$S_+\underbrace{S_-|-\rangle}_{0} = S^2|-\rangle - \underbrace{S_z^2|-\rangle}_{\frac{\hbar^2}{4}|-\rangle} + \underbrace{\hbar S_z|-\rangle}_{-\frac{\hbar^2}{2}|-\rangle}$$

because $|-\frac{3}{2}\rangle$ doesn't exist

$$S^2|-\rangle = \frac{3}{4}\hbar^2|-\rangle$$

$|\pm\rangle$ are both eigenstate of S^2

w/ eigenvalue $\frac{3}{4}\hbar^2$

(degenerate w.r.t. S^2 but not w.r.t S_z)

S_z and S^2 are compatible (there exists a complete set of states ...)

$[\hat{S}_z, \hat{S}^2] = 0$

Prove $[J_z, J^2] = 0$ directly using comm relations [HW]

$$S_-|+\rangle = c_-(\tfrac{1}{2})\,|-\rangle$$

How do we determine $c_-(\tfrac{1}{2})$?

$$\langle +|\underbrace{S_-^\dagger}_{S_+} = \langle -|\, c_-^*(\tfrac{1}{2})$$

Now combine

$$\langle +|S_+S_-|+\rangle = \langle -|\, c_-^*(\tfrac{1}{2})\, c_-(\tfrac{1}{2})\,|-\rangle = |c_-(\tfrac{1}{2})|^2$$

$$= \langle +|\, S^2 - S_z^2 + \hbar S_z\,|+\rangle$$

$$= \langle +|\, \frac{3\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 + \hbar\left(\frac{\hbar}{2}\right)|+\rangle$$

$$= \hbar^2 \underbrace{\langle +|+\rangle}_{1}$$

$$|c_-(\tfrac{1}{2})| = \hbar$$

$$\Rightarrow c_-(\tfrac{1}{2}) = \hbar e^{i\theta}$$

$$S_+|-\rangle = c_+(-\tfrac{1}{2})|+\rangle$$

$$\Rightarrow \quad c_+(-\tfrac{1}{2}) = \langle + | S_+ | - \rangle$$

$$= \langle - | S_+^\dagger | - \rangle^*$$

$$= \langle - | S_- | + \rangle^*$$

$$= \langle - | c_-(\tfrac{1}{2}) | - \rangle^*$$

$$= c_-^*(\tfrac{1}{2})$$

$$= \hbar e^{-i\theta}$$

Thus $\quad S_-|+\rangle = \hbar e^{i\theta}|-\rangle$

$\qquad\quad S_+|-\rangle = \hbar e^{-i\theta}|+\rangle$

Redefine $|-\rangle' = e^{i\theta}|-\rangle \quad \Rightarrow \quad S_-|+\rangle = \hbar|-\rangle'$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad S_+|-\rangle' = \hbar|+\rangle$

(still normalized + orthogonal to $|+\rangle$)

Now drop primes:

$$\boxed{\begin{aligned} S_-|+\rangle &= \hbar|-\rangle \\ S_+|-\rangle &= \hbar|+\rangle \end{aligned}}$$

[Alt, given $|+\rangle$, define $|-\rangle = \frac{1}{\hbar} S_-|+\rangle$

It is (1) an eigenstate
(2) normalized]

Recall: $\hat{\mathbb{1}} = |+\rangle\langle+| + |-\rangle\langle-|$

$\hat{S}_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-| \right)$ [problem set 7]

Claim $S_- = \hbar |-\rangle\langle+|$

$S_+ = \hbar |+\rangle\langle-| = S_-^{\dagger}$

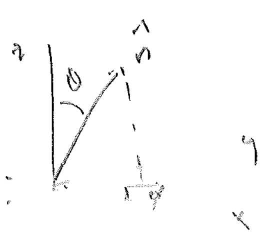
Prove by acting on an arbitrary ket $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

Then $S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+| \right)$

$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right)$

Let $\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

necessary??



Define $S_n = \hat{n}\cdot\vec{S} = \sin\theta\cos\phi\, S_x + \sin\theta\sin\phi\, S_y + \cos\theta\, S_z$

Now let's represent all these operators as matrices