optional

Two body problem

$$H = T + V$$

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$V = V(r_1 - r_2)$$

Let $\begin{cases} r_{cm} = \frac{m_1}{M} r_1 + \frac{m_2}{M} r_2 \\ r = r_1 - r_2 \end{cases}$ where $M = m_1 + m_2$

Then $\begin{cases} r_1 = r_{cm} + \frac{m_2}{M} r \\ r_2 = r_{cm} - \frac{m_1}{M} r \end{cases}$

$$T = \frac{1}{2} m_1 \left(\dot{r}_{cm} + \frac{m_2}{M} \dot{r}\right)^2 + \frac{1}{2} m_2 \left(\dot{r}_{cm} - \frac{m_1}{M} \dot{r}\right)^2$$

$$= \frac{1}{2}(m_1 + m_2) \dot{r}_{cm}^2 + \left(\text{zero cross term}\right) + \frac{1}{2} \underbrace{\frac{m_1 m_2 (m_2 + m_1)}{M^2}}\, \dot{r}^2$$

$$= \frac{1}{2} M \dot{r}_{cm}^2 + \frac{1}{2} \mu \dot{r}^2$$

$\frac{m_1 m_2}{M} \equiv \mu$, reduced mass

Now $p_{cm} = M \dot{r}_{cm} = p_1 + p_2 = p_{sys} \Rightarrow \dot{r}_{cm} = \frac{p_{cm}}{M}$

$p = \mu \dot{r} \Rightarrow \dot{r} = \frac{p}{\mu}$

$\mu \approx m_{\text{lighter particle}}$

$$H = \underbrace{\frac{p_{cm}^2}{2M}}_{\text{free particle}} + \underbrace{\frac{p^2}{2\mu} + V(r)}_{\text{reduced problem}}$$

$|E, l, m\rangle$ = simultaneous eigenstate of H, L^2, L_3

$$u_{Elm}(r,\theta,\phi) = R_{Elm}(r)\, Y_{lm}(\theta,\phi)$$

$$H\, u_{Elm} = E\, u_{Elm}$$

$$H = -\frac{\hbar^2}{2m_e}\nabla^2 + V(r)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{}$$

You will show this $= -\dfrac{L^2}{\hbar^2}$

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\right]R_{Elm}Y_{lm} + V(r)R_{Elm}Y_{lm} = E R_{Elm}Y_{lm}$$

$$-\frac{\hbar^2}{2m_e}\left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{Elm}}{\partial r}\right)} - \frac{l(l+1)}{r^2}R_{Elm}\right] + V(r)R_{Elm} = E R_{Elm}$$

$= \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}(r R_{Elm})$, easy to verify

$$\frac{d^2}{dr^2}(rR_{Elm}) - \frac{l(l+1)}{r^2}(rR_{Elm}) = -\frac{2m_e}{\hbar^2}(E - V(r))(rR_{Elm})$$

radial Schrödinger eqn

N.B. Coefficients of the eqn depend on l, but not m.
therefore eigenvalues E + eigenfunctions $R(r)$ do not
depend on m i.e. the $(2l+1)$ eigenfunctions

$$u_{Elm}(r,\theta,\phi) = R(r)\, Y_{lm}(\theta,\phi) \qquad m = -l, \ldots, l$$

are degenerate $\Rightarrow$ this is due to spherical symmetry

[Drop m from label for $R_{El}(r)$]

$$u_{Elm}(r,\theta,\phi) = R_{El}(r)\, Y_{lm}(\theta,\phi)$$

E only depends on l