Position state eigenstates of $\{H, L^2, L_3\}$

$$u(r, \theta, \phi) = \langle \vec{x} \mid E, l, m \rangle$$

↑ should be labelled by E, l, m

Eigenvalue eqns are just linear P.D.E.'s

Try separable ansatz $u(r, \theta, \phi) = R(r) \underbrace{Y_{lm}(\theta, \phi)}_{g(\theta) f(\phi)}$

$$L_3 \mid E, l, m \rangle = m\hbar \mid E, l, m \rangle$$

$$\frac{\hbar}{i} \frac{\partial}{\partial \phi} u(r, \theta, \phi) = m \hbar u$$

$$\frac{\hbar}{i} \frac{df}{d\phi} = m\hbar f$$

$$\frac{df}{f} = i m \, d\phi$$

$$\ln f = i m \phi + \text{const}$$

$$f = (\text{const}) \, e^{im\phi} \qquad \Rightarrow f(\phi + 2\pi) = e^{2\pi i m} f(\phi)$$

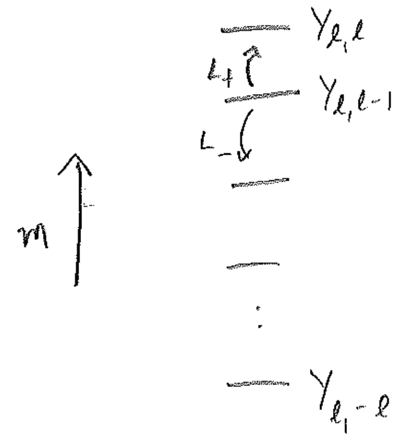
We know already that for any type of angular momentum m = integer or half-integer.

But for orbital angular momentum, m must be integer in order that $f(\phi + 2\pi) = f(\phi)$, ie single valued function of position

(otherwise $f(\phi + 2\pi) = -f(\phi)$)

$$\Rightarrow Y_{lm}(\theta, \phi) = g(\theta) \, e^{im\phi}, \qquad m = -l, \ldots, l$$

Recall $L_\pm = \hbar e^{\pm i\phi}\left(\pm \frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)$



$$L_+ Y_{\ell\ell} = 0$$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) g(\theta) e^{i\ell\phi} = 0$$

$$\frac{dg}{d\theta} + i\cot\theta\,(i\ell)\,g = 0$$

$$\frac{dg}{d\theta} = -g\,\ell\cot\theta$$

$$\frac{dg}{g} = -\ell\,\frac{\cos\theta\,d\theta}{\sin\theta}$$

$$\log g = -\ell\ln(\sin\theta) + \text{const}$$

$$g = C(\sin\theta)^\ell$$

$$Y_{\ell\ell}(\theta,\phi) = C\,(\sin\theta)^\ell e^{i\ell\phi}$$

[determine C later]

Recall $L_{\pm} Y_{lm} = \sqrt{(l \mp m)(l \pm m + 1)}\, \hbar\, Y_{l,m\pm 1}$

$$Y_{l,l-1} = \frac{1}{\sqrt{2l}} \frac{L_-}{\hbar} Y_{l,l}$$

$$= \frac{e^{-i\phi}}{\sqrt{2l}} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right) C (\sin\theta)^l e^{il\phi}$$

$$= \frac{e^{-i\phi} C}{\sqrt{2l}} \left(-l (\sin\theta)^{l-1} \cos\theta - l (\cos\theta)(\sin\theta)^{l-1} \right) e^{il\phi}$$

$$Y_{l,l-1} = -\sqrt{2l}\, C\, (\sin\theta)^{l-1} (\cos\theta)\, e^{i(l-1)\phi}$$

etc

$Y_{\ell m}(\theta, \phi)$ are normalized by

$$\int |Y_{\ell m}(\theta,\phi)|^2 \underbrace{d\Omega}_{\text{solid angle measure } = \sin\theta \, d\theta \, d\phi} = 1$$

For example

$$\int |Y_{\ell\ell}(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi$$

$$= |C|^2 \underbrace{\int_0^{\pi} (\sin\theta)^{2\ell+1} d\theta}_{\frac{2(2\ell)!!}{(2\ell+1)!!}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

Recall $n!! = n(n-2)(n-4)\ldots$

$$\Rightarrow C = \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!} \frac{1}{4\pi}} \cdot (-1)^{\ell}$$

standard convention

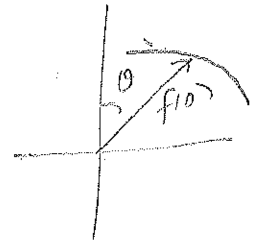
$$\left[\int_0^{\pi} \sin^{2\ell+1}\theta \, d\theta = \int_{-1}^{1} \underbrace{(1-x^2)^{\ell}}_{\sum_{m=0}^{\ell} (-)^m x^{2m} \cdot \frac{\ell!}{m!(\ell-m)!}} dx = 2 \sum_{m=0}^{\ell} (-1)^m \frac{\ell!}{m!(\ell-m)!} \frac{1}{(2m+1)} = \frac{2(2\ell)!!}{(2\ell+1)!!} \right]$$

	ℓ	
S. state	0	$Y_{00} = \frac{1}{\sqrt{4\pi}}$
P-state	1	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$
		$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$
		$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$
D-state	2	$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\phi}$
		$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$
		$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$
		$Y_{2,-1} = +\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{-i\phi}$
		$Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{-2i\phi}$

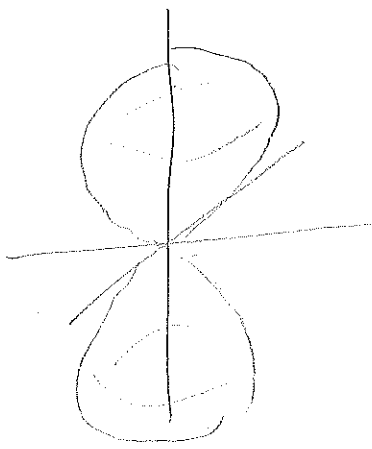
$$Y_{\ell,-m} = (-1)^m Y_{\ell,m}^*$$

[cf Griffiths (2e) 139]

visualize using polar ~~pot~~ plots
(explain)



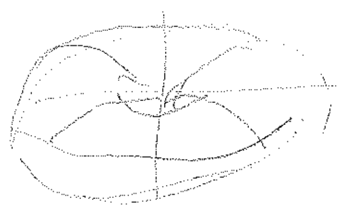
$|Y_{10}|^2 \sim \cos^2\theta$



← "p_z"

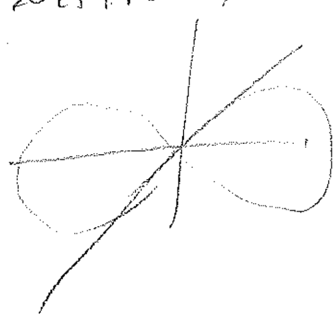
$|Y_{11}|^2 \sim \sin^2\theta$

$\sim |Y_{1,-1}|^2$



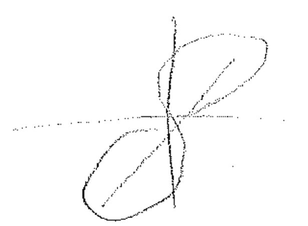
$Y_{1,1} - Y_{1,-1} \sim \sin\theta\,(e^{i\phi} + e^{-i\phi}) \approx 2\sin\theta\cos\phi$

$|Y_{11} - Y_{1,-1}|^2 \sim \sin^2\theta\cos^2\phi$



p_x

$|Y_{11} + Y_{1,-1}|^2 \sim \sin\theta\,\sin^2\phi$



p_y