

Angular momentum review

Recall that rotations are generated by angular momentum op. $\vec{J}$

$$U = e^{-\frac{i\alpha \hat{n} \cdot \vec{J}}{\hbar}} \quad \text{for a rotation through angle } \alpha \text{ about } \hat{n}.$$

We showed this implies

$$[J_1, J_2] = i\hbar J_3 \quad \text{plus cyclic perms}$$

$$\text{For a particle w/ spin, } \vec{J} = \vec{L} + \vec{S}$$

$$\text{For a particle w/o spin, } \vec{J} = \vec{L} \quad \text{so}$$

$$[L_i, S_j] = 0$$

$$[L_1, L_2] = i\hbar L_3 \quad \text{plus cyclic perms}$$

$$[S_1, S_2] = i\hbar S_3$$

$$\text{Classically } \vec{L} = \vec{r} \times \vec{p}$$

$$L_3 = x_1 p_2 - x_2 p_1$$

$\Downarrow$

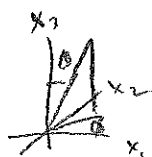
$$\text{Quantum: } \hat{L}_3 = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1$$

[Hw:] show using $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ that
 $[L_1, L_2] = i\hbar L_3$ and cyclic perms.

Orbital ang. mom. acts on position space wavefunctions.

In position space: $L_3 = \frac{\hbar}{i} (x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})$

Recall spherical coord:



$$x_3 = r \cos \theta$$

$$x_1 = r \sin \theta \cos \phi$$

$$x_2 = r \sin \theta \sin \phi$$

I claim: $L_3 = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$

First proof: chain rule $\frac{\partial}{\partial \phi} = \frac{\partial x_1}{\partial \phi} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \phi} \frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial \phi} \frac{\partial}{\partial x_3}$

$$= \underbrace{-r \sin \theta \sin \phi}_{x_2} \frac{\partial}{\partial x_1} + \underbrace{r \sin \theta \cos \phi}_{x_1} \frac{\partial}{\partial x_2} + 0$$

$$= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \quad \checkmark$$

Second proof: $U = e^{-i\alpha \frac{L_3}{\hbar}} = e^{-\alpha \frac{\partial}{\partial \phi}}$

$$U \psi(r, \theta, \phi) = e^{-\alpha \frac{\partial}{\partial \phi}} \psi(r, \theta, \phi)$$

$$= \left[1 - \alpha \frac{\partial \psi}{\partial \phi} + \frac{1}{2} (-\alpha)^2 \frac{\partial^2 \psi}{\partial \phi^2} + \dots \right]$$

$$= \psi(r, \theta, \phi - \alpha)$$

so U rotates wavefunction through angle α about x_3 -axis

[HW]: show that

$$L_1 = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right)$$

$$L_2 = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right)$$

This implies

$$L_{\pm} = L_1 \pm iL_2 = i\hbar \left(\underbrace{[\sin\phi \mp i\cos\phi]}_{\mp ie^{\pm i\phi}} \frac{\partial}{\partial\theta} + \cot\theta \underbrace{[\cos\phi \pm i\sin\phi]}_{e^{\pm i\phi}} \frac{\partial}{\partial\phi} \right)$$

$$\boxed{L_{\pm} = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right)}$$

Save this for later

Consider a particle in a spherically-symmetric potential $V(r)$ (a.k.a. central potential because $\vec{F} = -\vec{\nabla}V$ points toward or away from the origin)

$$H = \frac{p^2}{2m} + V(r) \rightarrow -\frac{\hbar^2}{2m} \nabla^2 + V(r)$$

In spherical coordinates

$$\nabla^2 = \underbrace{\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r})}_{\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}} + \frac{1}{r^2} \underbrace{\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}}_{\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta}}$$

Claim: spherical symmetry of potential $\Rightarrow [H, \vec{L}] = 0$

First consider $[H, L_3]$

$$\frac{\partial}{\partial \phi} \left(-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \frac{\partial}{\partial \phi}$$

$$\text{so } [H, L_3] = 0$$

Not so easy to see in spherical coordinates, but nevertheless true, that

$$[H, L_1] = 0$$

$$[H, L_2] = 0$$

(unless write H in cartesian coords, use $[H, L_3]$ plus cyclic perms to show the others)

more general argument:

H is invariant under rotations because

$V(r)$ only depends on r and ∇^2 is a scalar operator

Let $H|E\rangle = E|E\rangle$

Then $U|E\rangle$ is also an eigenstate w/ same energy E
(because rotating the state will produce another state with same energy)

$H(U|E\rangle) = E(U|E\rangle)$

$HU|E\rangle = UE|E\rangle = UH|E\rangle$

$(HU - UH)|E\rangle = 0$

$[H, U]|E\rangle = 0$

Because energy eigenstates are a complete basis

$[H, U]|\psi\rangle = 0$ for any $|\psi\rangle$

Because this holds for any $|\psi\rangle$

$[H, U] = 0$

$U = e^{-\frac{i\alpha \hat{n} \cdot \vec{J}}{\hbar}} = 1 - \frac{i\alpha}{\hbar} \hat{n} \cdot \vec{J} + \dots$

$\Rightarrow [H, \hat{n} \cdot \vec{J}] = 0$

$[H, \vec{J}_i] = 0$

Particle w/ spin

$[H, \vec{L}_i] = 0$

$[L_i, L_j] \neq 0$ so diff comps of orb-ang mom not compatible

However $[L^2, L_i] = 0$ where $L^2 = L_1^2 + L_2^2 + L_3^2$

[proved earlier in problem set]

$$\text{Also } [H, L^2] = \sum_i [H, L_i^2] = \sum_i ([H, L_i] L_i + L_i [H, L_i]) = 0$$

$\Rightarrow H, L^2$, and one component of $\vec{L}$ are mutually compatible

choose $L_3 \Rightarrow \{H, L^2, L_3\} =$ set of mutually commuting ops.

Goal: find simultaneous eigenstates $|E, l, m\rangle$

$$H |E, l, m\rangle = E |E, l, m\rangle$$

$$L^2 |E, l, m\rangle = \hbar^2 l(l+1) |E, l, m\rangle$$

$$L_3 |E, l, m\rangle = \hbar m |E, l, m\rangle$$

Note: because $[H, L_{\pm}] = 0$, $|E, l, m\rangle$ will be degenerate for all m