Particle in a 3d potential

$$H = \frac{p^2}{2m} + V(x_1, x_2, x_3)$$

$$p^2 = p_1^2 + p_2^2 + p_3^2 \longrightarrow -\hbar^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) = -\hbar^2 \underbrace{\nabla^2}_{\text{Laplacian}}$$

In position space

$$H \longrightarrow -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})$$

$$|E\rangle \longrightarrow u_E(\vec{x}) = \langle \vec{x} | E \rangle$$

$$H|E\rangle = E|E\rangle \longrightarrow \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right) u_E = E\, u_E$$

3d time-indep. Schrodinger eqn

Consider a 3d potential of the *special* form

$$V(x_1, x_2, x_3) = \tilde{V}(x_1) + \tilde{V}(x_2) + \tilde{V}(x_3)$$ → symmetric under $x_1 \leftrightarrow x_2$ (or any perm of x_1, x_2, x_3)

This includes

1) particle in a cube
2) isotropic harmonic oscillator
$$V(\vec{x}) = \frac{1}{2} k \, \vec{x} \cdot \vec{x}$$

Then

$$\sum_{i=1}^{3} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \tilde{V}(x_i) \right] u(\vec{x}) = E \, u(\vec{x})$$

Try a separable ansatz: $u(\vec{x}) = u^{(1)}(x_1)\, u^{(2)}(x_2)\, u^{(3)}(x_3)$

$$\sum \underbrace{\frac{1}{u^{(i)}} \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \tilde{V}(x_i) \right] u^{(i)}(x_i)}_{E^{(i)}} = E$$

$$E = E^{(1)} + E^{(2)} + E^{(3)}$$

$$\left[-\frac{\hbar^2}{2m} \frac{d^2 u^{(i)}}{dx^2} + \tilde{V}(x) \right] u^{(i)} = E^{(i)} u^{(i)}$$ ← 1d eqn, same eqn for all i

Let $u_n(x)$ be solutions of this eqn w/ eigenenergies E_n

For each i, choose one of these solutions

$$u(x_1, x_2, x_3) = u_{n_1}(x_1)\, u_{n_2}(x_2)\, u_{n_3}(x_3)$$
$$E = E_{n_1} + E_{n_2} + E_{n_3}$$

		E	degeneracy
gnd state	$u_1(x_1)\,u_1(x_2)\,u_1(x_3)$	$3E_1$	1
1^{st} excited	$u_2(x_1)\,u_1(x_2)\,u_1(x_3)$ $u_1(x_1)\,u_2(x_2)\,u_1(x_3)$ $u_1(x_1)\,u_1(x_2)\,u_2(x_3)$	$2E_1+E_2$	3
	not nec. in this order	E_1+2E_2	3
		$2E_1+E_3$	3
		$E_1+E_2+E_3$	6

Bound states in 1d. are always nondegenerate

In >1d, there are often degeneracies especially when potential has symmetry

[e.g.