

[After they've done the problem]

33-1

Harmonic oscillator summary

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

operator approach (Dirac)

$$\begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{i}{\sqrt{2\hbar m\omega}} p \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} x - \frac{i}{\sqrt{2\hbar m\omega}} p \end{cases}$$

$$\Rightarrow [a, a^\dagger] = 1$$

$$\Rightarrow H = \hbar\omega \left(N + \frac{1}{2}\right) \quad \text{where } N = a^\dagger a.$$

Let $|n\rangle$ be eigenstates of N w/ eigenvalue n
+ therefore eigenstates of H w/ eigenvalue $\hbar\omega(n + \frac{1}{2})$

You showed $a^\dagger + a$ are raising + lowering operators.

$$a|n\rangle = \sqrt{n} |n-1\rangle$$

$$a|n\rangle = \sqrt{n+1} |n+1\rangle$$

memmos: (larger integer)

$$\Rightarrow a|0\rangle = 0$$

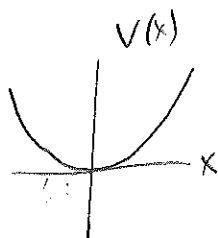
$$N|0\rangle = 0|0\rangle$$

$$H|0\rangle = \frac{1}{2}\hbar\omega |0\rangle$$

grd stl energy

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

Harmonic oscillator
(eg diatomic molecule)



$$E = \frac{p^2}{2m} + \frac{1}{2} k_s x^2$$

Natural frequency $\omega = \sqrt{\frac{k_s}{m}} \Rightarrow E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 = \hbar \omega \left[\frac{m \omega}{2 \hbar} \hat{x}^2 + \frac{1}{2 m \hbar \omega} \hat{p}^2 \right]$$

operator approach

Define $\hat{a} = \sqrt{\frac{m \omega}{2 \hbar}} \hat{x} + \frac{i}{\sqrt{2 m \hbar \omega}} \hat{p}$

non hermitian! (not an observable)

$$\hat{a}^\dagger = \sqrt{\frac{m \omega}{2 \hbar}} \hat{x} - \frac{i}{\sqrt{2 m \hbar \omega}} \hat{p}$$

$[\hat{a}, \hat{a}^\dagger] = 1$ using $[\hat{x}, \hat{p}] = i \hbar 1$

Define $\hat{N} = \hat{a}^\dagger \hat{a} = \frac{m \omega}{2 \hbar} \hat{x}^2 + \frac{1}{2 m \hbar \omega} \hat{p}^2 - \frac{1}{2} 1 = \frac{\hat{H}}{\hbar \omega} - \frac{1}{2} 1$

$$\hat{H} = \hbar \omega \left(\hat{N} + \frac{1}{2} 1 \right)$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \Rightarrow \hat{N} \hat{a}^\dagger = \hat{a}^\dagger (\hat{N} + 1)$$

$$[\hat{N}, \hat{a}] = -\hat{a} \Rightarrow \hat{N} \hat{a} = \hat{a} (\hat{N} - 1)$$

Let $\hat{N} |\lambda\rangle = \lambda |\lambda\rangle$ be eigenstates of $\hat{N}$ (normalized)

The $\hat{N} (\hat{a}^\dagger |\lambda\rangle) = (\lambda + 1) (\hat{a}^\dagger |\lambda\rangle)$ i.e.

$$\hat{N} (\hat{a} |\lambda\rangle) = (\lambda - 1) (\hat{a} |\lambda\rangle) \text{ i.e.}$$

$\hat{a}^\dagger |\lambda\rangle \sim |\lambda + 1\rangle$ raising
 $\hat{a} |\lambda\rangle \sim |\lambda - 1\rangle$ lowering

$$\begin{array}{c} \hat{a}^\dagger \uparrow |\lambda + 1\rangle \\ \hat{a} \downarrow |\lambda - 1\rangle \end{array}$$

not necessary
as done in problem

$|\lambda\rangle$ are also energy eigenstates

$$\hat{H}|\lambda\rangle = \underbrace{\hbar\omega(\lambda + \frac{1}{2})}_E |\lambda\rangle$$

From analysis of Schrodinger eqn $\Rightarrow E \geq V_{\min} = 0$
 $\Rightarrow \lambda \geq -\frac{1}{2}$

$$\exists \lambda_{\min} \Rightarrow a|\lambda_{\min}\rangle = 0$$

$$\langle \lambda_{\min} | a^\dagger = 0$$

$$0 = \langle \lambda_{\min} | a^\dagger a | \lambda_{\min} \rangle = \langle \lambda_{\min} | N | \lambda_{\min} \rangle = \lambda_{\min} = 0$$

$$E_{\min} = \hbar\omega(\lambda_{\min} + \frac{1}{2}) = \frac{1}{2}\hbar\omega \leftarrow \text{ground state energy of harmonic oscillator}$$

$|0\rangle = \text{ground state}$

All other states obtained by raising λ by 1 $\Rightarrow |n\rangle$
 $a^\dagger |1\rangle \sim |2\rangle$
 $n = \text{non-negative integers}$

Eigenstate $|n\rangle$ has energy $E = \hbar\omega(n + \frac{1}{2})$

one can show:

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

Mnemonic = $\sqrt{\text{larger}}$

$$|1\rangle = a^\dagger |0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}} a^\dagger |1\rangle = \frac{1}{\sqrt{2}} a^{\dagger 2} |0\rangle$$

$$|3\rangle = \frac{1}{\sqrt{3}} a^\dagger |2\rangle$$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

Harmonic oscillator in position space

33-2

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} x + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx}$$

$$= \frac{1}{\sqrt{2}} \left(y + \frac{d}{dy} \right)$$

$$y = \sqrt{\frac{m\omega}{\hbar}} x$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(y - \frac{d}{dy} \right)$$

Ground state setup

$$a |0\rangle = 0$$

In position space

$$\left(y + \frac{d}{dy} \right) u_0(y) = 0$$

$$\frac{du_0}{dy} = -y u_0$$

$$\frac{du_0}{u_0} = -y dy$$

$$\ln u_0 = -\frac{1}{2} y^2 + \text{const}$$

$$u_0 = C e^{-\frac{1}{2} y^2}$$

$$= C e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\text{Normalized: } u_0 = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

In prob 2a

$$u_n = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}} \left[y - \frac{d}{dy} \right] \right)^n \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{1}{2}y^2}$$

$$= \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{n!} 2^n} \underbrace{\left(y - \frac{d}{dy} \right)^n e^{-\frac{1}{2}y^2}}_{\text{this is } H_n(y) e^{-\frac{1}{2}y^2}}$$

$$\text{eg } \left(y - \frac{d}{dy} \right) e^{-\frac{1}{2}y^2}$$

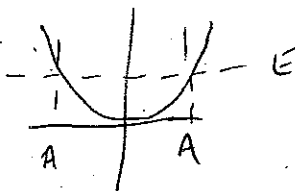
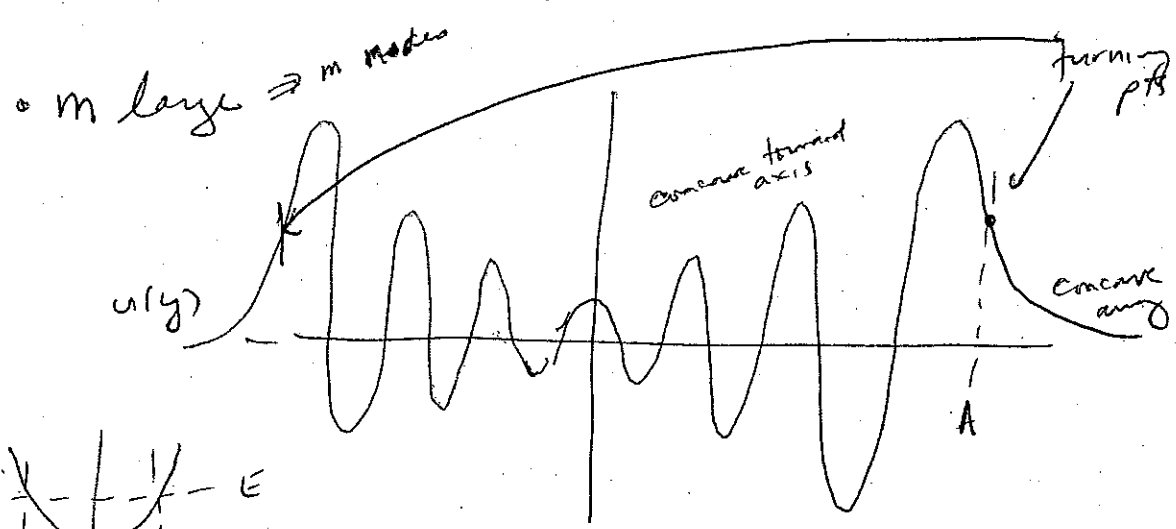
$$= \left[y - (-y) \right] e^{-\frac{1}{2}y^2} = 2y e^{-\frac{1}{2}y^2}$$

$$H_1(y) = 2y$$

n	$H_n(y)$
0	1
1	$2y$
2	$4y^2 - 2$
3	$8y^3 + 2y$
4	$16y^4 - 48y^2 + 12$

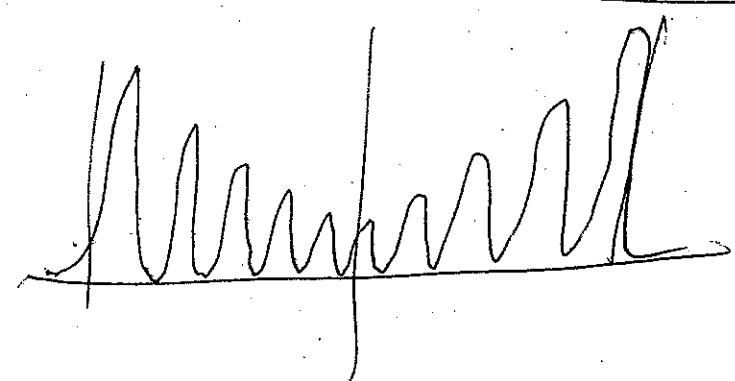
$$u_n = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{n!} 2^n} H_n(y) e^{-\frac{1}{2}y^2}$$

33-4
~~14~~
 16
 18



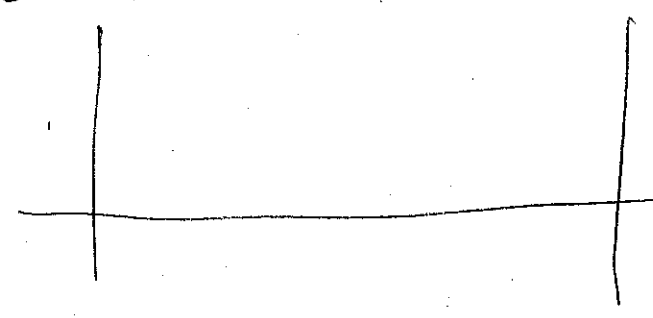
~~With~~ beyond turning points, $|u|^2 \neq 0$

$|u|^2$



Why is it more probable to find ~~the~~ particle near turning points? Because particle moves ~~slow~~ more slowly there!

~~Wave function is real and so $|u|^2$ is real~~



Sharp

Prob $\sim$ time spent $\sim \frac{1}{v}$

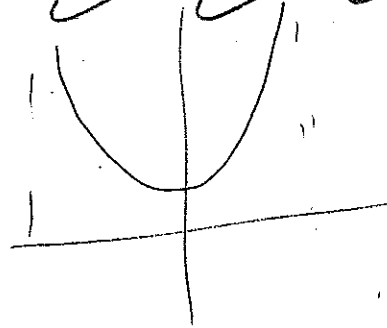
point of time spent
 between x & $x + \Delta x$
 $\Delta t = \frac{\Delta x}{v}$

~~$\psi = A \cos \omega t$~~
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yes

33-7

$P(x) \sim \frac{1}{v} \sim \frac{1}{\sqrt{A^2 - x^2}}$

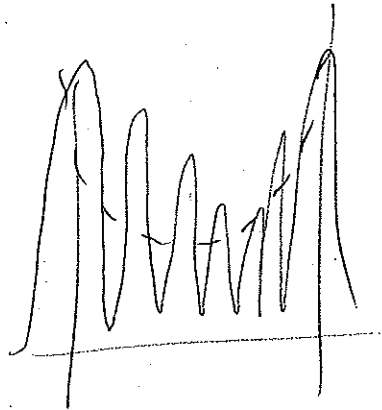


$\frac{1}{2} m v^2 = E - V(x) = \frac{1}{2} m \omega^2 (A^2 - x^2)$

$v = \pm \omega \sqrt{A^2 - x^2}$

do we
 have a problem?

Qm



Correspondence principle

$Qm \rightarrow CM$ as $n \rightarrow \infty$

The power of the operator approach

$$\langle n | x^2 | n \rangle = \int dx \quad \cancel{H_n(x)}^2 x^2 |u_n(x)|^2$$

$$\sim \int dy \quad y^2 H_n^2(y) e^{-y^2}$$

difficult for $n = 143!$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

$$\langle n | x^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | (a + a^\dagger)(a + a^\dagger) | n \rangle$$

$$\underbrace{\langle n | a^2 | n \rangle}_0 + \underbrace{\langle n | a a^\dagger | n \rangle}_{\substack{\sqrt{n} \sqrt{n-1} | n-2 \rangle \\ n-1}} + \underbrace{\langle n | a^\dagger a | n \rangle}_{\substack{\sqrt{n+1} | n \rangle \\ n+1}} + \underbrace{\langle n | a^\dagger a^\dagger | n \rangle}_n$$

$$= \frac{\hbar}{2m\omega} (2n+1)$$

Problem: show $\langle n | x | n \rangle = 0$
 $\langle n | p | n \rangle = 0$

In 2010, I ~~clashed~~
~~with~~ talked with
 Qu'ran of some ppl
~~about~~
 → a x.r. pt energy
 → photons

(also from: $\{a, a^\dagger\} = 1$
 $a^2 = (a^\dagger)^2 = 0$)