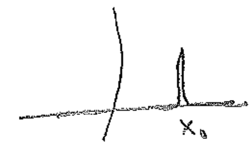
Review

observable | operator | eigenstate | in position space; $\psi(x) = \langle x|\psi\rangle$

$x \qquad \hat{x} \qquad |x_0\rangle \qquad \langle x|x_0\rangle = \delta(x-x_0)$

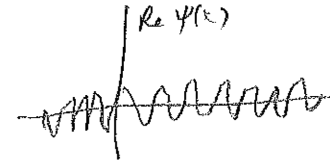
$\Delta x = 0$



not normaliz.

$p \qquad \hat{p} \qquad |p\rangle \qquad \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$\Delta p = 0$



not norm.

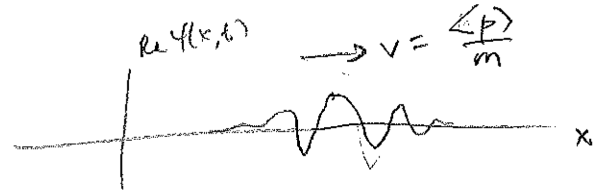
Normalizable states obtained by linear combinations of eigenstates (wavepackets)

$|\psi\rangle = \int dp\, \phi(p)\, |p\rangle$

normalizable

For a free particle, $\psi(x,t)$ moves at constant speed + spreads in time

$E = \frac{p^2}{2m}$



What happens if particle is subjected to a conservative force?

$$F_x = -\frac{dV}{dx}$$

$$E = \frac{p^2}{2m} + V(x)$$

$\therefore$ energy eigenstates are <u>no longer</u> momentum eigenstates.

need first find energy eigenvalues & eigenfunctions

Particle in a potential

$$E = T + V = \frac{p^2}{2m} + V(x)$$ ↙ non rel.

[recall Bohr model]

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x)$$

[What is $\hat{V}$?]

Power series expansion of $V(x) = V_0 + V_1 x + V_2 x^2 + \dots$

$$\Rightarrow \hat{V} = V_0 + V_1 \hat{x} + V_2 \hat{x}^2 + \dots$$

$\hat{p}$, $\hat{x}$ are hermitian

$\hat{V}$ is hermitian if $V(x)$ is a real function, ie $V_0, V_1, \dots$ are real

$$\hat{V}^\dagger = V_0^* + V_1^* \hat{x}^\dagger + V_2^* (\hat{x}^2)^\dagger + \dots$$

But $(\hat{x}^n)^\dagger = \hat{x}^\dagger \hat{x}^\dagger \dots \hat{x}^\dagger = \hat{x}^n$

$\Rightarrow \hat{V}^\dagger = \hat{V}$ if V

$\Rightarrow \hat{H}$ is hermitian

classically, Newton's 2nd law $\frac{dp}{dt} = F_x = -\frac{dV}{dx}$

Recall Ehrenfest theorem for rate of change of expectation value

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle$$

Let $A = p$
$$[p,H] = [p, \frac{p^2}{2m} + V(x)]$$
$$= [p, V(x)]$$
$$= -i\hbar \frac{dV}{dx}$$

(from HW problem)

or can evaluate it in position space

$[p, V]_{pos} \psi(x)$
$= \frac{\hbar}{i}\frac{d}{dx}(V(x)\psi) - V\frac{\hbar}{i}\frac{d}{dx}\psi$
$= -i\hbar\frac{dV}{dx}$

$$\Rightarrow \frac{d}{dt}\langle p\rangle = -\left\langle \frac{dV}{dx}\right\rangle$$

HW: for free particle

$$\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\langle [x,H]\rangle$$
$$= \frac{1}{i\hbar}\left\langle \frac{1}{m}[x,p]\,p\right\rangle$$
$$= \frac{1}{m}\langle p\rangle$$
$$\frac{d}{dt}\langle p\rangle = 0$$

check that these are obeyed by gaussian wavepacket

Energy eigenvalue eqn

$$\hat{H}|E\rangle = E|E\rangle$$

$$\left(\frac{\hat{P}^2}{2m} + \hat{V}\right)|E\rangle = E|E\rangle$$

Evaluate this in position space

In position space, energy eigenstates are $\langle x|E\rangle \equiv u_E(x)$

$u(x)$ is commonly used for energy eigenstate

$$(\hat{H})_{pos} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right) + V(x)$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

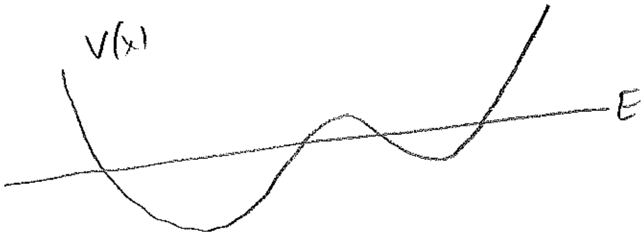
$\Rightarrow$ energy eigenfunctions satisfy

$$-\frac{\hbar^2}{2m}\frac{d^2 u_E}{dx^2} + V(x)\,u_E = E\,u_E \qquad (\text{t.i. S.e})$$

$$\frac{1}{u_E}\frac{d^2 u_E}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)$$

Qualitative solutions of t.i.s.e

t.i.s.e. $\Rightarrow$ $$\boxed{\frac{1}{u_E}\frac{d^2 u_E}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)}$$

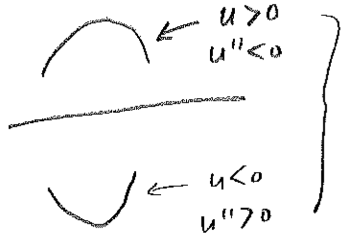


classically allowed regions $\Rightarrow E - V(x) > 0 \Rightarrow \frac{u''}{u} < 0 \Rightarrow$ concave toward x-axis

classically forbidden regions $\Rightarrow E - V(x) < 0 \Rightarrow \frac{u''}{u} > 0 \Rightarrow$ concave ~~away~~ away from axis

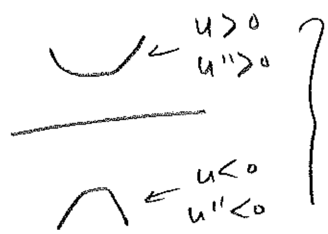
turning point $\Rightarrow E - V(x) = 0 \Rightarrow \frac{u''}{u} = 0 \Rightarrow$ point of inflection

concave toward axis



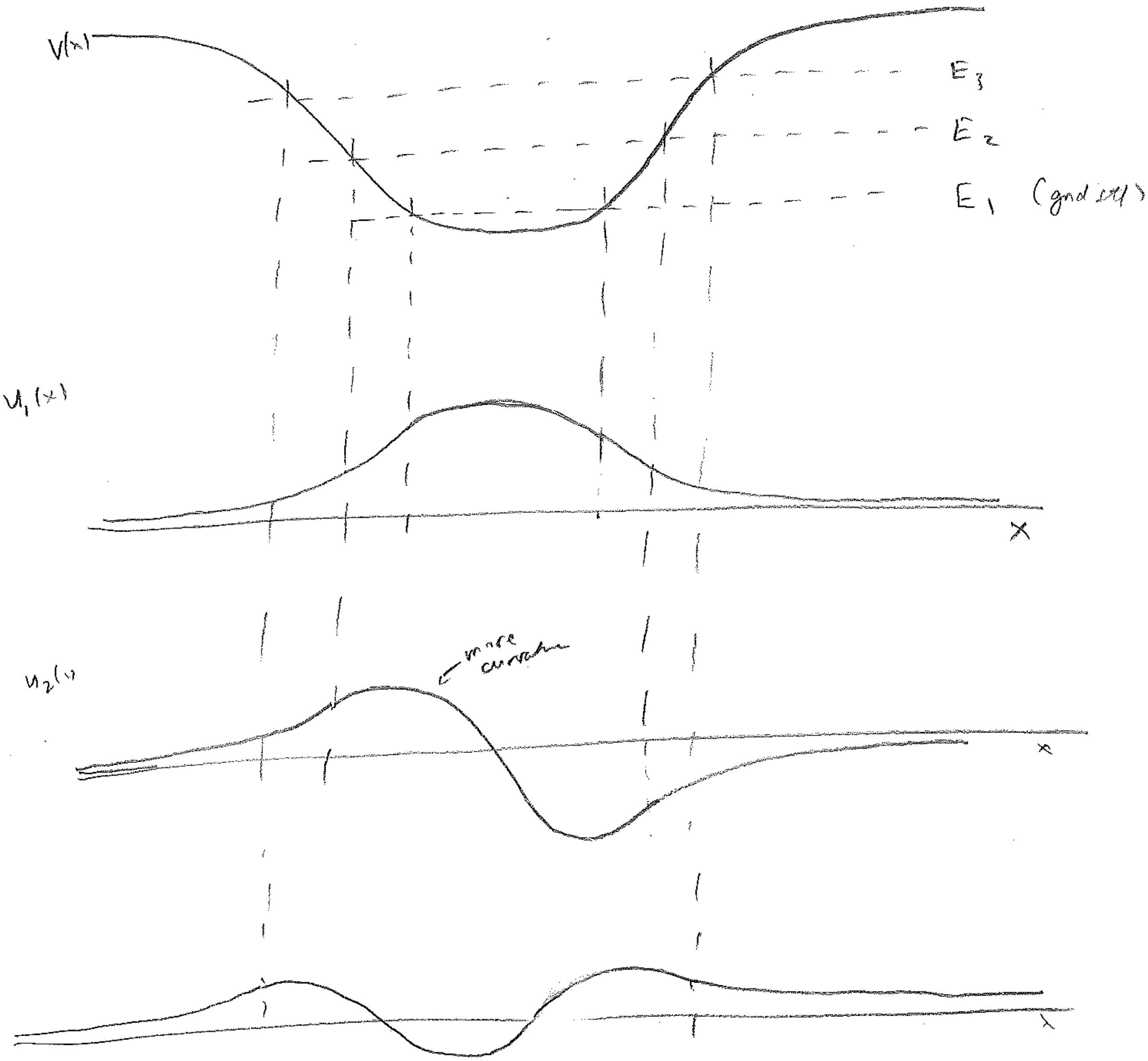
$\frac{u''}{u} < 0$

concave away from axis



$\frac{u''}{u} > 0$

Suppose the potential has the shape



For energies between the eigenvalues, $u(x) \not\to 0$ as $x \to \infty$

Suppose $E < V_{min}$ $\Rightarrow$ all regions forbidden

$\Rightarrow$ always curve away $\Rightarrow$ $u(x) \not\to 0$
not normalizable

$\Rightarrow$ $\exists$ no energy eigenvalue $\forall E < V_{min}$

Recall: position space

$$\psi(x,t) = \langle x | \psi(t)\rangle \Rightarrow |\psi(t)\rangle = \int dx\, \psi(x,t)\, |x\rangle$$

momentum space

$$\phi(p,t) = \langle p | \psi(t)\rangle \rightarrow |\psi(t)\rangle = \int dp\, \phi(p,t)\, |p\rangle$$

energy space (assume eigenvalues discrete)

$$c_n(t) = \langle E_n | \psi(t)\rangle \Rightarrow |\psi(t)\rangle = \sum_n c_n(t)\, |E_n\rangle$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle = \sum c_n(0)\, e^{-\frac{iHt}{\hbar}} |E_n\rangle = \sum c_n(0)\, e^{-\frac{iE_nt}{\hbar}} |E_n\rangle$$

so $c_n(t) = c_n(0)\, e^{-\frac{iE_nt}{\hbar}}$

Then

$$\psi(x,t) = \langle x | \psi(t)\rangle = \sum_n c_n(0)\, e^{-\frac{iE_nt}{\hbar}} \langle x | E_n\rangle$$

Let $u_n(x) = \langle x | E_n\rangle$ = energy eigenstate in position space

$$\boxed{\psi(x,t) = \sum c_n(0)\, e^{-\frac{iE_nt}{\hbar}} u_n(x)}$$

where $c_n(0) = \langle E_n | \psi(0)\rangle = \int dx\, \underbrace{\langle E_n | x\rangle}_{u_n^*(x)} \underbrace{\langle x | \psi(0)\rangle}_{\psi(x,0)}$

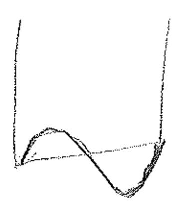
$$\boxed{c_n(0) = \int dx\, u_n^*(x)\, \psi(x,0)}$$

Review of particle in a box

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = Eu \quad \text{for} \quad 0<x<L$$

$$u(x) = 0 \quad \text{for} \quad \begin{cases} x<0 \\ x>L \end{cases}$$

$$\begin{cases} u_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right) \\ E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2} \end{cases}$$



- concave toward x-axis
- continuous u
- $\frac{du}{dx}$ discontinuous only where $V \to \infty$