Time evolution of a free particle

Schrodinger eqn

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H \, |\psi(t)\rangle$$

or equivalently

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$$

For a free particle $E = K \overset{\text{nonrel}}{=} \frac{1}{2} m v^2 = \frac{p^2}{2m}$

$$\Rightarrow \hat{H} = \frac{\hat{p}^2}{2m}$$

$\Rightarrow$ momentum eigenstates are also energy eigenstates

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

$$\hat{H}\,|p\rangle = \frac{p^2}{2m}\,|p\rangle$$

[www.eng.fsu.edu/~dommelen/ φ demo of gp vs phase velocity]

Expand initial state in momentum basis

$$|\psi(0)\rangle = \int dp \;\; \phi(p,0)\, |p\rangle$$

$$|\psi(t)\rangle = \int dp \;\; \phi(p,0)\, e^{-\frac{iHt}{\hbar}} |p\rangle$$

$$= \int dp \;\; \underbrace{\phi(p,0)\, e^{-\frac{ip^2 t}{2m\hbar}}}_{\phi(p,t)} \; |p\rangle$$

$$\text{momentum probability density} = |\phi(p,t)|^2 = \left|\phi(p,0)\, e^{-\frac{ip^2t}{2m\hbar}}\right|^2 = |\phi(p,0)|^2$$

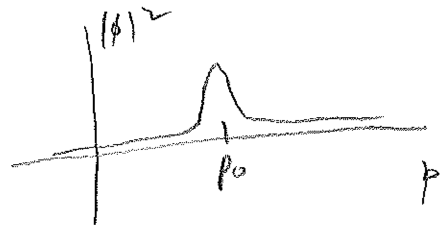
$\hookrightarrow$ doesn't change in time

Position space wavefunction:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \;\; \phi(p,t)\, e^{\frac{ipx}{\hbar}} = \underbrace{\frac{1}{\sqrt{2\pi\hbar}} \int dp \;\; \phi(p,0)\, e^{i\left(\frac{px}{\hbar} - \frac{p^2 t}{2m\hbar}\right)}}_{\text{linear comb of travelling waves ("wave packet")}}$$

Consider an initial gaussian momentum distribution

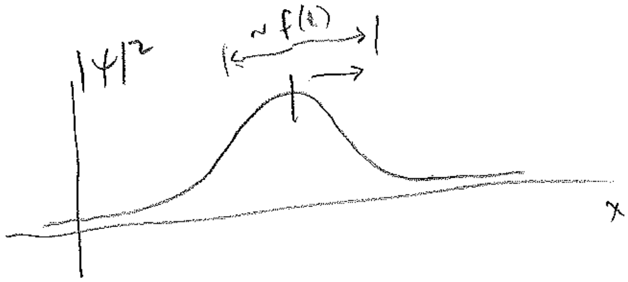
$$\phi(p,0) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$



In HW, you will show that position space wavefunction evolves ~~...~~ to $\Psi(x,t)$ ~~that~~ such that

$$|\Psi(x,t)|^2 = \left(\frac{1}{2\pi f^2(t)}\right)^{\frac{1}{2}} e^{-\frac{(x-\frac{p_0}{m}t)^2}{2f^2(t)}} \qquad [\text{HW}]$$

You will calc $f(t)$ in HW, but $f(t)$ grows monotonically w/ t



$\langle x \rangle = \frac{p_0}{m} t$

Peak moves w/ speed $\frac{p_0}{m}$. $\Rightarrow$ not a stationary state because of interference bw energy eigenstates

gaussian spreads out w/ time (position becomes more uncertain)

[HW: ~~show~~ derive $\frac{d}{dt}\langle x \rangle = \frac{\langle p \rangle}{m}$; show that it is satisfied here.]

QM. w/ Mathematica