Momentum space wavefunction

Recall $|\psi\rangle = \int dx\ |x\rangle\langle x|\psi\rangle = \int dx\ \underbrace{\psi(x)}\ |x\rangle$

position space wavefunction
(probability amplitude)

Similarly $|\psi\rangle = \int dp\ |p\rangle\langle p|\psi\rangle = \int dp\ \underbrace{\phi(p)}\ |p\rangle$

momentum space wavefunction

How are $\psi(x)$ & $\phi(p)$ related?

$$\psi(x) = \langle x|\psi\rangle = \int dp\ \langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\ e^{\frac{ipx}{\hbar}}\ \phi(p)$$

$$\phi(p) = \langle p|\psi\rangle = \int dx\ \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\ e^{-\frac{ipx}{\hbar}}\ \psi(x)$$

They are Fourier transforms of each other.

Quick review

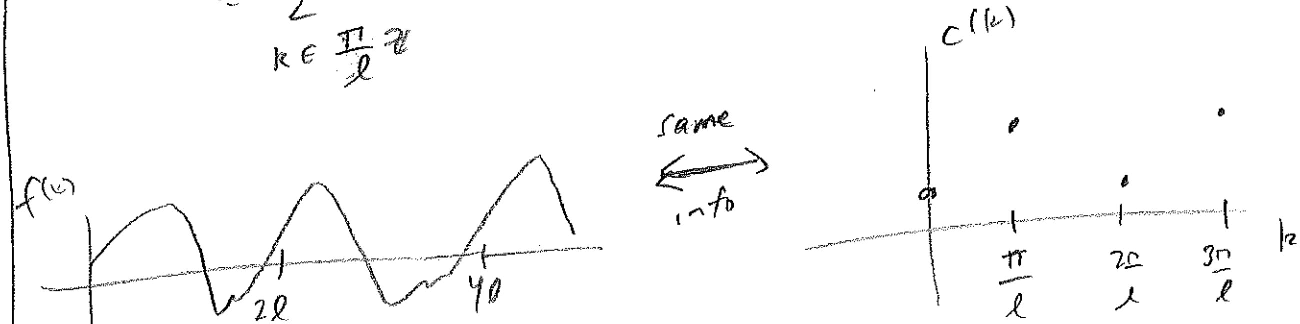
Skip unless asked

Phys 300 I didn't do this anymore

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{\ell}\right) + a_n \cos\left(\frac{n\pi x}{\ell}\right) \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{\ell}} \qquad \text{Let } k = \frac{n\pi x}{\ell}$$

$$= \sum_{k \in \frac{\pi}{\ell}\mathbb{Z}} c(k) e^{ikx}$$



double period ⇒ more dense (more information!)

Recall $\langle\psi|\psi\rangle = 1 \Rightarrow \int dx\, |\psi(x)|^2 = 1 \Rightarrow \psi$ is square-integrable (in position space)

Normalized state

$$1 = \langle\psi|\psi\rangle = \int dp \langle\psi|p\rangle\langle p|\psi\rangle = \int dp\, \phi^*(p)\,\phi(p) = \int dp\, |\phi(p)|^2$$

$\Rightarrow \phi(p)$ is square-integrable (in momentum space)

$|\phi(p)|^2$ = probability density in momentum space

i.e. $|\phi(p)|^2 dp$ = prob. of measuring momentum between p & $p+dp$

$$\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle = \int dp\, \underbrace{\langle\psi|\hat{p}|p\rangle}_{p\langle\psi|p\rangle}\langle p|\psi\rangle = \int dp\; p\, |\phi(p)|^2$$

Let's Fourier transform

$$\langle p\rangle = \int dp\; \phi^*(p)\; p\left[\frac{1}{\sqrt{2\pi\hbar}}\int dx\, e^{-ipx/\hbar}\,\psi(x)\right]$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int dp\; \phi^*(p) \int dx\; \underbrace{p\, e^{-ipx/\hbar}}_{-\frac{\hbar}{i}\frac{d}{dx}e^{-ipx/\hbar}}\; \psi(x)$$

IBP: $\begin{cases} u = \psi(x) \\ du = \frac{d\psi}{dx}\,dx \end{cases}$ $\quad dv = -\frac{\hbar}{i}\frac{d}{dx}\left(e^{-ipx/\hbar}\right)dx \quad v = -\frac{\hbar}{i}e^{-ipx/\hbar}$

$$\underbrace{-\frac{\hbar}{i}\psi(x)e^{-ipx/\hbar}\Big|_{-\infty}^{\infty}}_{0} + \int dx\; e^{-ipx/\hbar}\,\frac{\hbar}{i}\frac{d\psi}{dx}$$

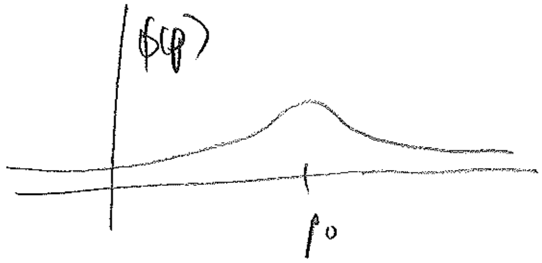
$$\Rightarrow \int dx\; \frac{\hbar}{i}\frac{d\psi}{dx}\;\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int dp\; \phi^*(p)\, e^{-ipx/\hbar}}_{\psi^*(x)}$$

$$\langle p\rangle = \int_{-\infty}^{\infty} dx\; \psi^*(x)\underbrace{\left(\frac{\hbar}{i}\frac{d}{dx}\right)}_{\hat{p}_{pos.}}\psi(x)$$

[$|p\rangle$ is an idealization ; perfectly well-defined p] → infinitely spread out in position space
try for something more realistic : range of momenta] 29-4

gaussian

Consider a particle with momentum space distribution $\phi(p) = N e^{-\beta(p-p_0)^2}$



[expect to measure p_0, with some spread]

$$1 = \int dp\, |\phi(p)|^2 = N^2 \int_{-\infty}^{\infty} dp\, e^{-2\beta(p-p_0)^2} = N^2 \underbrace{\int_{-\infty}^{\infty} dp'\, e^{-2\beta(p')^2}}_{I(\beta)}$$

$$\Rightarrow N = \frac{1}{\sqrt{I(\beta)}}$$

$$I = \int_{-\infty}^{\infty} dx\, e^{-2\beta x^2}$$

$$I^2 = \int_{-\infty}^{\infty} dx\, e^{-2\beta x^2} \int_{-\infty}^{\infty} dy\, e^{-2\beta y^2} = \int \underbrace{dx\,dy}_{r\,dr\,d\theta}\, e^{-2\beta \underbrace{(x^2+y^2)}_{r^2}}$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\infty} r e^{-2\beta r^2} dr}_{\frac{1}{4\beta}\underbrace{\int_0^{\infty} du\, e^{-u}}_{1}}$$

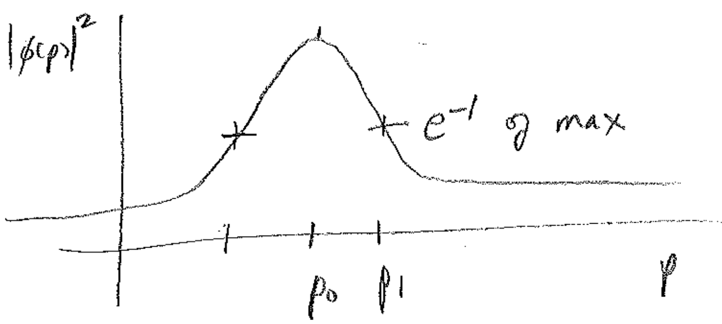
$u = 2\beta r^2$
$du = 4\beta r\, dr$

$$= \frac{\pi}{2\beta}$$

$$I = \sqrt{\frac{\pi}{2\beta}} \quad \Rightarrow \quad \phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$

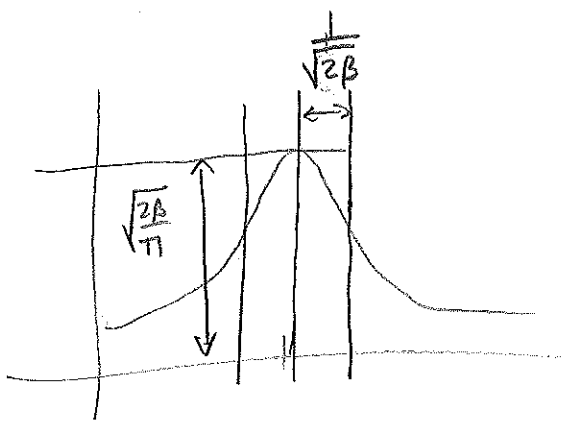
$$|\phi(p)|^2 = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} e^{-2\beta(p-p_0)^2}$$

$$I(\beta) = \int dp\, e^{-2\beta p^2} = \sqrt{\frac{\pi}{2\beta}}$$



$$\Rightarrow \quad 2\beta(p_1-p_0)^2 = 1$$

$$|p_1-p_0| = \frac{1}{\sqrt{2\beta}}$$



$\beta \to \infty \Rightarrow$ well defined p.

$$\langle p\rangle = \int dp\ |\phi(p)|^2 p$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp\ e^{-2\beta(p-p_0)^2}\ p \qquad \text{Let } p = p_0 + p'$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\ e^{-2\beta p'^2} (p_0 + p')$$

$$= p_0 \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\ e^{-2\beta p'^2} + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} dp'\ e^{-2\beta p'^2} p'}_{\text{odd integrand}}$$

$$= p_0$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dp\ |\phi(p)|^2 p^2$$

$$= \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dp'\ e^{-2\beta p'^2} (p_0^2 + 2p_0 p' + p'^2)$$

$$= p_0^2 + 0 + \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} dp'\ e^{-2\beta p'^2} p'^2}$$

$$I(\beta) = \int dp'\ e^{-2\beta p'^2} = \sqrt{\frac{\pi}{2\beta}} \longrightarrow -\frac{1}{2}\frac{dI}{d\beta} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\ \frac{1}{\beta^{3/2}}$$

$$\langle p^2\rangle = p_0^2 + \frac{1}{4\beta}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{1}{2\sqrt{\beta}}$$

[problem: $\langle p^{2n}\rangle$]

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{\frac{ipx}{\hbar}} \phi(p)$$

what is the position space wavefunction of a gaussian momentum distrib.?

$$\phi(p) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2} \quad \Rightarrow \Delta p = \frac{1}{2\sqrt{\beta}}$$

If don't shift, then time dependent case becomes harder

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \underbrace{\int dp \; e^{-\beta(p-p_0)^2 + \frac{ipx}{\hbar}}}_{e^{\frac{ip_0 x}{\hbar}} \int dq \; e^{-\beta q^2 + \frac{iqx}{\hbar}}}$$

Let $q = p - p_0 \Rightarrow p = q + p_0$

[see ahead for how to do this w/o shifting]

Do this first

Consider $\displaystyle\int_{-\infty}^{\infty} dq \; e^{-(aq^2 + bq)}$

$$= \int dq \; e^{-a\left(q + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}}$$

$$= e^{\frac{b^2}{4a}} \int dq' \; e^{-aq'^2}$$

$$= \sqrt{\frac{\pi}{a}} \; e^{b^2/4a}$$

$$\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{\frac{1}{4\beta}\left(\frac{ix}{\hbar}\right)^2} e^{\frac{ip_0 x}{\hbar}}$$

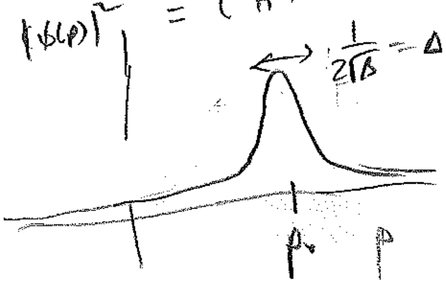
$$= \left(\frac{1}{2\pi\hbar^2\beta}\right)^{\frac{1}{4}} e^{-\frac{x^2}{4\beta\hbar^2}} e^{\frac{ip_0 x}{\hbar}}$$

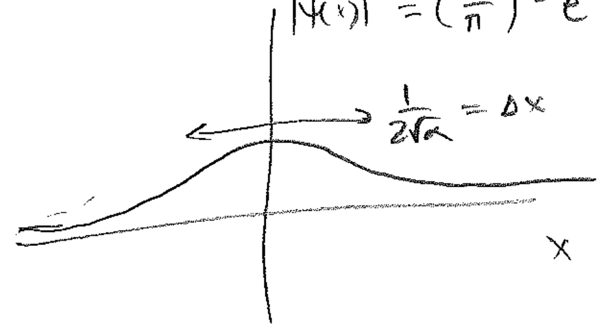
$$= \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} e^{\frac{ip_0 x}{\hbar}} \quad \text{where } \alpha = \frac{1}{4\beta\hbar^2}$$

Fourier tr of a gaussian is a gaussian (times a phase)

$$|\phi(p)|^2 = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} e^{-2\beta(p-p_0)^2}$$

$$|\psi(x)|^2 = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} e^{-2\alpha x^2}$$





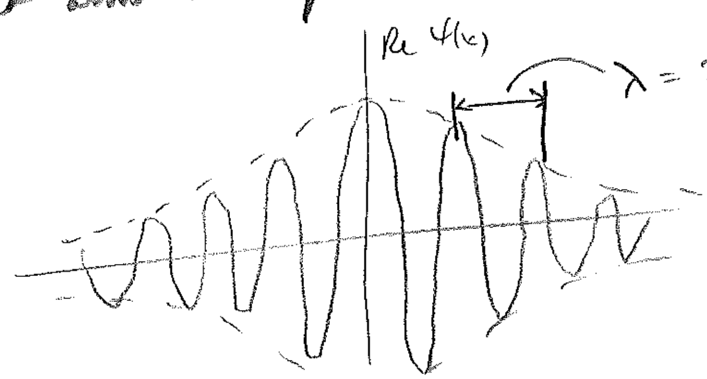
Since $\alpha\beta = \frac{1}{4\hbar^2}$

$$\left(\frac{1}{2\sqrt{\alpha}}\right)\left(\frac{1}{2\sqrt{\beta}}\right) = \frac{\hbar}{2}$$ widths are inverse related

Products of ~~widths~~ uncertainties

$$\Delta x \, \Delta p = \frac{\hbar}{2}$$ Heisenberg is saturated

What about the phase? $\mathrm{Re}\,\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2} \cos\left(\frac{p_0 x}{\hbar}\right)$



$\lambda = 2\pi\left(\frac{\hbar}{p_0}\right) = \frac{h}{p_0}$

= de Broglie wavelength of particle w/ mom p_0

[wavelength not perfectly well defined because momentum not " "]