Momentum eigenstates

[Besides being a generator of translations]

Momentum is an observable

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

p = possible result of a measurement of momentum = any real #

$|p\rangle$ = state with perfectly well-defined momentum (idealization)

Completeness $\quad \int dp\, |p\rangle\langle p| = \mathbb{1}$

orthonormality $\quad \langle p'|p\rangle = \delta(p-p')$

$$|p\rangle = \int dx \; |x\rangle \underbrace{\langle x|p\rangle}$$

$\psi_p(x)$ = wavefunction of momentum eigenstate in position space (ie in X-basis)

$$P\,|p\rangle = p\,|p\rangle$$

$$\Downarrow$$

$$(P)_{pos.}\, \psi_p(x) = p\, \psi_p(x)$$

$$\frac{\hbar}{i}\frac{d\psi_p}{dx} = p\,\psi_p$$

$$\frac{d\psi_p}{\psi_p} = \frac{ip}{\hbar}\,dx$$

$$\ln \psi_p = \frac{ipx}{\hbar} + \text{const}$$

$$\psi_p = A\,e^{\frac{ipx}{\hbar}}$$

Thus $\langle x|p\rangle = A\,e^{\frac{ipx}{\hbar}}$ (A determined later)

Observe: like $|x\rangle$, $|p\rangle$ is not normalizable because $\psi_p(x)$ is not square integrable

$$\int_{-\infty}^{\infty} dx\, |\psi_p(x)|^2 = \int_{-\infty}^{\infty} dx\, |A|^2 = \infty \quad \text{(unless } A=0)$$

Probability density $|\psi_p(x)|^2 = |A|^2 = \text{const}$

equally likely to be anywhere in space!

$$\Delta x \geq \frac{\hbar}{2\Delta p} = \infty$$

$|p\rangle$ form an orthonormal basis in the Dirac sense

ie, $\langle p'|p\rangle = \delta(p'-p)$
Use this to determine A.

$$\delta(p-p') = \int_{-\infty}^{\infty} dx\, \langle p'|x\rangle\langle x|p\rangle$$

$$= \int_{-\infty}^{\infty} dx\; A^* e^{-\frac{ip'x}{\hbar}} \; A\, e^{\frac{ipx}{\hbar}}$$

$$= |A|^2 \int_{-\infty}^{\infty} dx\; e^{\frac{i(p-p')x}{\hbar}}$$

[Can "see" that r.h.s $= 0$ if $p \neq p'$ because cos + i sin = area vanishes.
But need to choose particular $|A|^2$ for it to be normalized

$$= |A|^2 \lim_{L\to\infty} \underbrace{\int_{-L}^{L} dx\; e^{\frac{i(p-p')x}{\hbar}}}_{\frac{\hbar}{i(p-p')} \underbrace{e^{\frac{i(p-p')x}{\hbar}}\Big|_{-L}^{L}}_{2i\sin\frac{(p-p')L}{\hbar}}}$$

$$\delta(p-p') = 2\hbar |A|^2 \lim_{L\to\infty} \frac{\sin\frac{(p-p')L}{\hbar}}{(p-p')}$$

Integrate both sides w.r.t. p

$$\int_{-\infty}^{\infty} \delta(p-p')\,dp = 2\hbar |A|^2 \lim_{L\to\infty} \underbrace{\int_{-\infty}^{\infty} dp \frac{1}{(p-p')} \sin\left[\frac{(p-p')L}{\hbar}\right]}_{\underbrace{\int_{-\infty}^{\infty} dy \frac{\sin y}{y}}_{\pi}}$$

Let $y = \frac{(p-p')L}{\hbar}$

(see next page)

$$1 = 2\pi\hbar |A|^2$$

$$A = \frac{1}{\sqrt{2\pi\hbar}} \quad \Rightarrow \quad \delta(p-p') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx\, e^{\frac{i(p-p')x}{\hbar}}$$

$$\Rightarrow \boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}}$$

$$|p\rangle = \int dx\, |x\rangle\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{\frac{ipx}{\hbar}} |x\rangle$$

$$|x\rangle = \int dp\, |p\rangle\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{-\frac{ipx}{\hbar}} |p\rangle$$

[change of basis]

unitary transformation

exam question

$$U = e^{-\frac{ia\hat{p}}{\hbar}}$$

$$U|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{-\frac{ipx}{\hbar}} e^{-\frac{ia\hat{p}}{\hbar}} |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{-\frac{ip(x+a)}{\hbar}} |p\rangle = |x+a\rangle$$