Momentum

Angular momentum $\vec{J}$ generates rotations:
a rotation through $\hat{n}\phi$ is implemented by unitary operator $U = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi}$

Hamiltonian H (energy) generates time translation
time evolution by t is implemented by $U = e^{-\frac{i}{\hbar}Ht}$

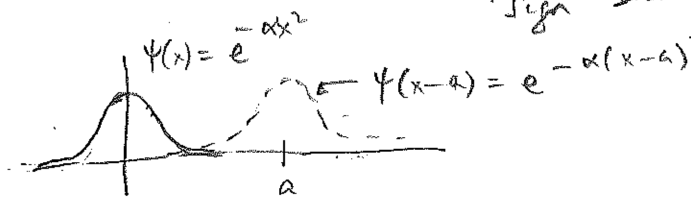
Momentum $\vec{P}$ generates spatial translations
translation by $\vec{a}$ is implemented by $U = e^{-\frac{i}{\hbar}\vec{P}\cdot\vec{a}}$

Eg in one dimension: $U|x\rangle = |x+a\rangle$

How does this act on a wavefunction?

$$U_{pos.}\,\psi(x) = \psi(x-a)$$

sign seems wrong but is actually correct



Let $|\psi\rangle = \int dx\ \psi(x)\,|x\rangle$

$$\begin{aligned} U|\psi\rangle &= \int dx\ \psi(x)\ U|x\rangle \\ &= \int dx\ \psi(x)\,|x+a\rangle \quad \} \ y = x+a \\ &= \int dy\ \psi(y-a)\,|y\rangle \quad \} \text{ rename } y\to x \\ &= \int dx\ \psi(x-a)\,|x\rangle \end{aligned}$$

Thus $U_{pos.}\,\psi(x) = \psi(x-a)$

$$\psi(x-a) = \psi(x) - a\frac{d\psi}{dx} + \frac{1}{2}a^2\frac{d^2\psi}{dx^2} - \frac{1}{3!}a^3\frac{d^3\psi}{dx^3} + \dots$$

$$= e^{-a\frac{d}{dx}}\psi(x)$$

Thus $U_{pos}\,\psi(x) = \psi(x-a)$

$$e^{-\frac{i}{\hbar}p_{pos}a}\,\psi(x) = e^{-a\frac{d}{dx}}\,\psi(x)$$

$$\Rightarrow \boxed{p_{pos} = \frac{\hbar}{i}\frac{d}{dx}} \qquad p_{pos}\,\psi(x) = \frac{\hbar}{i}\frac{d\psi}{dx}$$

$$\boxed{X_{pos} = x} \qquad X_{pos}\,\psi(x) = x\,\psi$$

Consider $[X, P]_{pos}\, \psi(x) = x\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi - \frac{\hbar}{i}\underbrace{\frac{d}{dx}(x\psi)}_{\psi + x\frac{d\psi}{dx}}$

$$= i\hbar\,\psi$$

Since this is valid for arbitrary ψ,

$$[x, p]_{pos} = i\hbar$$

In general

$$[\hat{X}, \hat{P}] = i\hbar\,\mathbb{1}$$ Heisenberg commutation relation

$\hat{X}, \hat{P}$ are incompatible

Recall generalized uncertainty principle

$$\Delta A\ \Delta B \geq \frac{1}{2}\left|\langle\psi|\,[A,B]\,|\psi\rangle\right|$$

$$\Delta x\ \Delta p \geq \frac{1}{2}\underbrace{\left|\langle\psi|\,[x,p]\,|\psi\rangle\right|}_{|i\hbar\,\langle\psi|\psi\rangle|}$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

Prob: $[x, p^n]$
$[x^n, p]$
$[V(x), p] = i\hbar\, V'(x)$