[Position observable]

Review [on side board]

observable A $\rightarrow$ hermitian operator $\hat{A}$

For observables of discrete spectra

eigenvalue eqn $\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$ finite or infinite

eigenvalues : results of possible measurements of A

eigenstates : state of determinate A

orthonormality $\langle a_n | a_m \rangle = \delta_{nm}$

~~completeness~~ $\sum_n |a_n\rangle\langle a_n| = 1$

decomposition of an arbitrary state into eigenstates

$$|\psi\rangle = \sum |a_n\rangle\langle a_n|\psi\rangle = \sum \psi_n |a_n\rangle$$

$\psi_n = \langle a_n|\psi\rangle$ = prob. amplitude

How does $\hat{A}$ act on $|\psi\rangle$?

$$\hat{A}|\psi\rangle = \sum \hat{A}\,|a_n\rangle\langle a_n|\psi\rangle = \sum a_n\psi_n\,|a_n\rangle$$

$|\psi\rangle$ is normalized

$$\Rightarrow 1 = \langle\psi|\psi\rangle = \sum \langle\psi|a_n\rangle\langle a_n|\psi\rangle = \sum \psi_n^*\psi_n = \sum|\psi_n|^2$$

Expectation value

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \sum a_n\psi_n \underbrace{\langle\psi|a_n\rangle}_{\psi_n^*} = \sum a_n|\psi_n|^2$$

$\Rightarrow |\psi_n|^2$ = prob. of measuring a_n

Position observable

position operator $\hat{x}$

$$\hat{x}|x\rangle = x|x\rangle$$

eigenvalue x = possible result of a measurement of position

= any real number

spectrum of $\hat{x}$ is $\mathbb{R}$

$|x\rangle$ = state of perfectly well-defined position (unphysical idealization)

Completeness [since all values of x are allowed, $\Sigma \to \int$]

$$\int dx\, |x\rangle\langle x| = \mathbb{1}$$

Decompose The $|\psi\rangle = \int dx\, |x\rangle\langle x|\psi\rangle = \int dx\, \psi(x)\, |x\rangle$ state indeterminate posit

where $\psi(x) = \langle x|\psi\rangle$ = a number that depends on ψ and x

ie a function of x!

= probability amplitude

= position-space wavefunction

[How to think of $\psi(x)$ as a vector: $\int dx\, \psi(x)|x\rangle \sim \sum_{n=-\infty}^{\infty} \psi(n)|x=n\rangle \Rightarrow \begin{pmatrix} \vdots \\ \psi(-1) \\ \psi(0) \\ \psi(1) \\ \vdots \end{pmatrix}$

Suppose $|\psi\rangle$ is normalized

$$1 = \langle\psi|\psi\rangle = \int dx \ \langle\psi|x\rangle\langle x|\psi\rangle$$

$$= \int dx \ \psi^*(x) \ \psi(x)$$

$$= \int dx \ |\psi(x)|^2$$

$$\left[\sum_n \psi^*(n)\psi(n) = (\cdots \psi^*(0)\,\psi^*(1) \cdots) \begin{pmatrix} \vdots \\ \psi(0) \\ \psi(1) \\ \vdots \end{pmatrix} \right.$$

$|\psi(x)|^2$ = probability <u>density</u> of obtaining x

A fcn that obeys $\int dx \, |\psi(x)|^2 < \infty$ called "square integrable" or "normalizable"

$$\psi(x) \xrightarrow[x \to \pm\infty]{} 0 \quad \text{faster than} \quad \frac{1}{\sqrt{|x|}}$$

How does $\hat{X}$ act on a state $|\psi\rangle$?

Let $|\psi\rangle = \int dx \;\; \psi(x)\, |x\rangle$

Then $\hat{X}|\psi\rangle = \int dx \;\; \psi(x)\; \hat{X}|x\rangle = \int dx \;\; x\,\psi(x)\, |x\rangle$

	abstract		position space	
so if	$\vert\psi\rangle$	$\longrightarrow$	$\psi(x)$	
then	$\hat{X}\vert\psi\rangle$	$\longrightarrow$	$x\,\psi(x)$	(multiply ψ by x)
ie	$\hat{X}$	$\longrightarrow$	$(\hat{X})_{pos} = x$	

Expectation value

$$
\begin{aligned}
\langle x\rangle &= \langle\psi|\,\hat{X}\,|\psi\rangle \\
&= \langle\psi| \int dx \;\; x\,\psi(x)\;|x\rangle \\
&= \int dx \;\underbrace{\langle\psi|x\rangle}_{\psi^*(x)}\; x\,\psi(x) \\
&= \int dx \;\; \psi^*(x)\; x \;\; \psi(x) \qquad\qquad [\text{as in } 2140] \\
&= \int dx \;\; x\,\underbrace{|\psi(x)|^2}_{\text{prob. density}}
\end{aligned}
$$

For self: $\hat{X} = \int dx\, dx'\, |x\rangle \underbrace{\langle x|\,X\,|x'\rangle}_{x\,\delta(x-x')} \langle x'|$

$= \int dx\, |x\rangle\, x\, \langle x|$

ie diagonal

What about orthonormality?

What is $\langle x' | x \rangle$? Expect $= 0$ if $x \neq x'$

Expect $= 1$? if $x = x'$

no!

Because x is a continuous variable $\langle x | x \rangle = \infty$

but ∞ is a subtle concept so let's be more careful

Consider an observable A w/ discrete eigenvalues a_n

$|\psi\rangle = \sum \psi_n |a_n\rangle$

$\psi_n = \langle a_n | \psi \rangle = \sum_m \langle a_n | a_m \rangle \langle a_m | \psi \rangle = \sum \delta_{nm} \psi_m$ ✓

↑ Kronecker delta

Consider an observable X w/ continuous eigenvalue x

$|\psi\rangle = \int dx' \, \psi(x') \, |x'\rangle$

$\psi(x') = \langle x' | \psi \rangle = \int dx \, \langle x' | x \rangle \langle x | \psi \rangle = \int dx \, \underbrace{\langle x' | x \rangle}_{\delta(x - x')} \psi(x)$

↑ Dirac delta "fcn"

(continuum analog of δ_{mn})

What is this Dirac delta "function" $\delta(x)$?

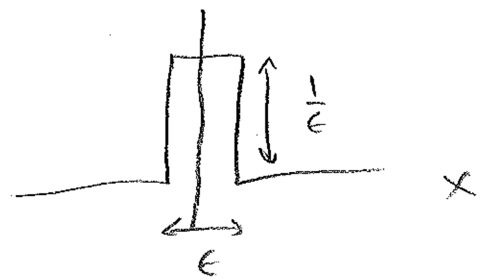
$\delta(x)$ is *defined* by two properties:

① $\delta(x) = 0$ if $x \neq 0$

② $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$

$\delta(x)$ is usually represented as a limit of a family of functions, for example

$$\delta(x) = \lim_{\epsilon \to 0} R_\epsilon(x) \quad \text{where} \quad R_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & |x| < \frac{1}{2}\epsilon \\ 0, & |x| > \frac{1}{2}\epsilon \end{cases}$$



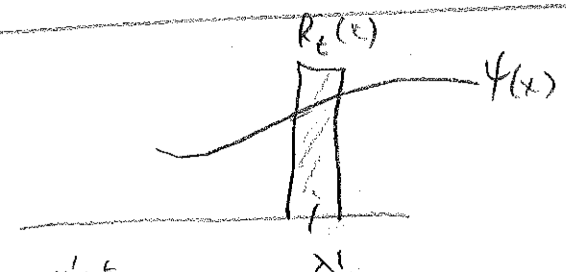
check:

① let $x = a \neq 0$. then $R_\epsilon(a) = 0$ provided $\epsilon < 2a$.
Since $\epsilon \to 0$ is smaller than any $a \neq 0$, $\delta(a) = 0$

② $\int_{-\infty}^{\infty} R_\epsilon(x)\, dx = \frac{1}{\epsilon} \cdot \epsilon = 1$ for any ϵ, so $\lim_{\epsilon \to 0} \int R_\epsilon(x)\, dx = 1$

Consider

$$\int_{-\infty}^{\infty} R_\epsilon(x - x')\psi(x)\, dx =$$



$$= \frac{1}{\epsilon} \int_{x' - \frac{\epsilon}{2}}^{x' + \frac{\epsilon}{2}} \psi(x)\, dx \approx \psi(x') \cdot \frac{1}{\epsilon} \int dx = \psi(x')$$

for small ϵ

$$\therefore \int_{-\infty}^{\infty} \delta(x - x')\psi(x)\, dx = \psi(x')$$

But this is precisely what we need for $\langle x' | x \rangle$

so $\boxed{\langle x' | x \rangle = \delta(x - x')}$

To prove: $\delta(x)$ is an even function

Need to establish that $\delta(-x)$ obeys the defining properties

① $\delta(-x) = 0$ if $x \neq 0$ ✓

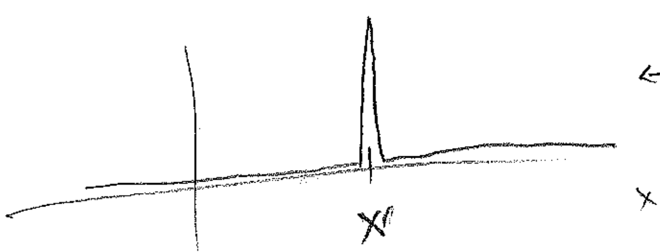
② $\displaystyle\int_{x=-\infty}^{x=\infty} dx\, \delta(-x) = \int_{y=\infty}^{y=-\infty} (-dy)\, \delta(y) = \int_{y=-\infty}^{\infty} dy\, \delta(y) = 1$ ✓

$y = -x$
$dy = -dx$

$\therefore \langle x'|x\rangle = \delta(x-x') = \delta(x'-x) = \langle x|x'\rangle$

$\langle x|\psi\rangle = \psi(x)$ is position space wavefunction of $|\psi\rangle$

$\Rightarrow \langle x|x'\rangle = \delta(x-x')$ is the position space wavefunction of $|x'\rangle$



← particle is perfectly localized

But $\delta(x-x')$ is not square integrable!

$$\int_{-\infty}^{\infty} dx\, \delta(x-x')\,\delta(x-x') = \delta(0) = \infty\,!$$

$|x\rangle$ not normalizable