

Precession of spin in a magnetic field

Classical analysis

$$E = -\vec{\mu} \cdot \vec{B}$$

$$\text{torque } \vec{\tau} = \vec{\mu} \times \vec{B}$$

compass needle will swing back & forth
+ come to rest (due to friction)
but spin will precess about mag field axis

$$\text{Recall for circulating charge: } \vec{\mu} = \left(\frac{q}{2m_e}\right) \vec{L}$$

$\checkmark$ gyromagnetic ratio for electron

$$\frac{d\vec{p}}{dt} = \vec{F}; \quad \frac{d\vec{J}}{dt} = \vec{\tau} \quad (\text{classically})$$

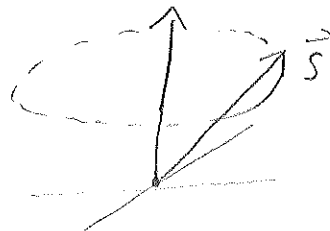
For intrinsic angular momentum (spin)

$$\text{we postulate that } \vec{\mu} = g \frac{q}{2m} \vec{S}$$

g determined by experiment
electrons: $q = -e$, $g \approx 2$
protons: $q = +e$, $g \approx 5.6$

$$\frac{d\vec{S}}{dt} = \frac{gq}{2m} \vec{S} \times \vec{B} \quad (\text{gyroscopic eqn})$$

$$\text{Let } \vec{B} = B \hat{z}$$



$$\begin{cases} \frac{dS_z}{dt} = 0 \end{cases} \Rightarrow S_z = \text{const}$$

$$\begin{cases} \frac{dS_x}{dt} = \omega_L S_y \\ \frac{dS_y}{dt} = -\omega_L S_x \end{cases} \quad \text{where } \omega_L = \frac{gqB}{2m} = \text{Larmor frequency}$$

S_z conserved,
 S_x, S_y not

$$\Rightarrow \frac{d^2 S_x}{dt^2} + \omega_L^2 S_x = 0 \Rightarrow \begin{cases} S_x = S_x(0) \cos(\omega_L t) \\ S_y = -S_x(0) \sin(\omega_L t) \\ S_z = S_z(0) \end{cases}$$

$$\text{Larmor period } T = \frac{2\pi}{\omega_L}$$

clockwise rotation

quantum analysis

$$E = -\vec{\mu} \cdot \vec{B} = -\left(\frac{gqB}{2m}\right) S_z = -\omega_L S_z$$

$$\Rightarrow \hat{H} = -\omega_L \hat{S}_z$$

H, S_z compatible $\Rightarrow$ share eigenstates $\nearrow S_z$ conserved

$$H|\pm\rangle_z = \mp \frac{\hbar\omega_L}{2} |\pm\rangle_z \quad \text{ie} \quad E_{\pm} = \mp \frac{\hbar\omega_L}{2}$$

Initial state

$$|\psi(0)\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle = \alpha e^{-\frac{iE_+t}{\hbar}} |+\rangle_z + \beta e^{-\frac{iE_-t}{\hbar}} |-\rangle_z$$

$$= \alpha e^{+i\frac{\omega_L t}{2}} |+\rangle_z + \beta e^{-i\frac{\omega_L t}{2}} |-\rangle_z$$

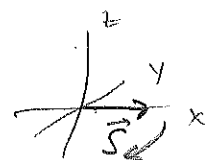
$$\text{Spin } \psi(t) = \begin{pmatrix} \alpha e^{i\frac{\omega_L t}{2}} \\ \beta e^{-i\frac{\omega_L t}{2}} \end{pmatrix} = e^{i\frac{\omega_L t}{2}} \begin{pmatrix} \alpha \\ \beta e^{-i\omega_L t} \end{pmatrix}$$

NB. $[H, S_z] = 0 \Rightarrow \langle S_z \rangle$ conserved

$[H, S_x] \neq 0 \Rightarrow \langle S_x \rangle \langle S_y \rangle$ not -

$[H, S_y] \neq 0$

Suppose initial spin points along +x axis



expect precession clockwise around z-axis

period $T = \frac{2\pi}{\omega_L}$

$$|\psi(0)\rangle = |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$$

Then $|\psi(t)\rangle = e^{\frac{i\omega_L t}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\omega_L t} \end{pmatrix}$

After $\frac{1}{4}$ period: $\omega_L t = \frac{\omega_L T}{4} = \frac{\pi}{2}$ $|\psi(\frac{T}{4})\rangle = e^{\frac{i\pi}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \sim |-y\rangle$
 $e^{-i\omega_L t} = e^{-\frac{i\pi}{2}} = -i$

After $\frac{1}{2}$ period $\omega_L t = \pi$ $|\psi(\frac{T}{2})\rangle = e^{\frac{i\pi}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sim |-x\rangle$
 $e^{-i\omega_L t} = -1$

After 1 period $\omega_L t = 2\pi$ $|\psi(T)\rangle = e^{i\pi} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim |+\rangle$
 -1

can detect this using nuclear interference



time for 1 complete rotation

$$e^{-\frac{iHt}{\hbar}} = e^{\frac{i\omega_L t}{\hbar} S_z} = e^{-\frac{i}{\hbar} (-\omega_L t) S_z}$$

rotates angle about z

Problem: calc $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$ via Heisenberg eqns...
 show same as classical eqns