

[if timing is bad, could move 15-1 up to here]

Time evolution of the state vector

Let the system be described by state $|\psi(0)\rangle$ at $t=0$.

Can we predict that state $|\psi(t)\rangle$ at time t ?

Let $U(t)$ be a unitary operator that implements translations in time

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \quad \Rightarrow \quad \langle\psi(t)| = \langle\psi(0)| U^\dagger(t)$$

U must be unitary so that probability is conserved
i.e. so that a state vector remains normalized

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)| \underbrace{U^\dagger U}_{=1} |\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$\mathbb{I}$ if U is unitary

Generalize to $U(t_2, t_1)$ where $|\psi(t_2)\rangle = U(t_2, t_1) |\psi(t_1)\rangle$

Satisfies composition property

$$|\psi(t_3)\rangle = U(t_3, t_2) |\psi(t_2)\rangle = U(t_3, t_2) U(t_2, t_1) |\psi(t_1)\rangle$$

$$U(t_3, t_1) = U(t_3, t_2) U(t_2, t_1)$$

We need to find a unitary operator $U(t)$ that obeys the composition property, and satisfied initial condition $U(0) = \mathbb{I}$

In CM, invariance under time translations

implies existence of conserved quantity, energy (Noether's theorem)

Claim: the operator associated w/ energy,
the Hamiltonian H , generates time translations, i.e.

$$U(t_2, t_1) = e^{-\frac{iH}{\hbar}(t_2 - t_1)} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{iH}{\hbar}(t_2 - t_1)\right)^n \mathbb{1}$$

• Check: $U(t, t) = e^0 = \mathbb{1}$

• Check: satisfies composition

$$e^{-\frac{iH}{\hbar}(t_3 - t_1)} = e^{-\frac{iH}{\hbar}(t_3 - t_2)} e^{-\frac{iH}{\hbar}(t_2 - t_1)}$$

Since $e^{\alpha A} e^{\beta B} = e^{\alpha A + \beta B}$ provided $[A, B] = 0$

$$[\text{Hint: prove this}] \quad e^{\alpha A} e^{\beta A} = e^{(\alpha + \beta)A}$$

• Check: unitary

$$U^\dagger(t_2, t_1) = \sum \frac{1}{n!} \left(-\frac{iH}{\hbar}(t_2 - t_1)\right)^{n\dagger}$$

$$= \sum \frac{1}{n!} \left(+i\frac{H^\dagger}{\hbar}(t_2 - t_1)\right)^n$$

$$= \sum \frac{1}{n!} \left(+i\frac{H}{\hbar}(t_2 - t_1)\right)^n \quad \text{by } H^\dagger = H$$

$$= e^{i\frac{H}{\hbar}(t_2 - t_1)}$$

The $U^\dagger(t_2, t_1) U(t_2, t_1) = e^{i\frac{H}{\hbar}(t_2 - t_1)} e^{-\frac{iH}{\hbar}(t_2 - t_1)} = e^0 = \mathbb{1}$

Claim: $U(t) = U(t, 0)$ satisfies:

$$\boxed{\frac{dU}{dt} = \frac{1}{i\hbar} H U}$$

Proof: $U(t+dt, 0) = U(t+dt, t) U(t, 0)$
 $= e^{-\frac{i}{\hbar} H dt} U(t, 0)$

$$\approx \left(1 - \frac{i}{\hbar} H dt\right) U(t, 0)$$

$$U(t+dt) - U(t) = \left(\frac{1}{i\hbar} H dt\right) U(t)$$

$$\frac{dU}{dt} = \lim_{dt \rightarrow 0} \frac{U(t+dt) - U(t)}{dt} = \frac{1}{i\hbar} H U(t) \quad QED$$

This implies the time dependent Schrodinger eqn.

$$\begin{aligned} \frac{d}{dt} |\psi(t)\rangle &= \frac{d}{dt} U(t) |\psi(0)\rangle \\ &= \frac{1}{i\hbar} H U(t) |\psi(0)\rangle \\ &= \frac{1}{i\hbar} H |\psi(t)\rangle \end{aligned}$$

$$\boxed{i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle}$$

How does the expectation value ^{of an observable A} change in time?

$$\langle A \rangle = \langle \psi(t) | A | \psi(t) \rangle = \langle \psi(0) | U^\dagger(t) A U(t) | \psi(0) \rangle$$

changes in
time
because
state changes

↑ states carry time dependence
operator generally not time dependent

$$\frac{d}{dt} \langle A \rangle = \langle \psi(0) | U^\dagger(t) A \frac{dU}{dt} | \psi(0) \rangle$$

$$\frac{1}{i\hbar} H U(t)$$

$$+ \langle \psi(0) | \frac{dU^\dagger}{dt} A U(t) | \psi(0) \rangle$$

$$- \frac{1}{i\hbar} U^\dagger(t) H$$

$$= \frac{1}{i\hbar} \langle \psi(t) | A H - H A | \psi(t) \rangle$$

$$\boxed{\frac{d}{dt} \langle A \rangle = \frac{1}{i\hbar} \langle \psi(t) | [A, H] | \psi(t) \rangle}$$

Ehrenfest's theorem

→ if $[A, H] = 0$ then $\langle A \rangle$ does not change over time

Suppose we have some operator A that commutes w/ H

$$[A, H] = 0$$

Then A also commutes w/ $U(t)$

Proof: $AH = HA$

$$AH^n = HAH^{n-1} = H^2AH^{n-2} = \dots = H^nA$$

$$\text{Since } U = \sum \frac{1}{n!} \left(-\frac{iHt}{\hbar}\right)^n \Rightarrow AU = UA$$

Suppose an initial state $|\psi(0)\rangle$ has a well-defined value of A
 i.e. $[A, H] = 0$.

$$A|\psi(0)\rangle = a|\psi(0)\rangle$$

Then if $[A, H] = 0$
 then the state into which it evolves has the same well-defined value

$$A|\psi(t)\rangle = AU|\psi(0)\rangle = UA|\psi(0)\rangle = Ua|\psi(0)\rangle = a|\psi(t)\rangle$$

i.e. it provides a "good quantum number" to characterize the state
 (one which does not change in time)

Since $[H, H] = 0$, the energy is a "good quantum no." (e.g. hydrogen)

Even if $|\psi(0)\rangle$ does not have a well-defined value of A
 the H expectation value will not change in time

if $[A, H] = 0$ (due to Ehrenfest).

$$\frac{d}{dt} \langle A \rangle = \frac{i}{\hbar} \langle [A, H] \rangle$$

$\Rightarrow A$ is a conserved quantity.

Given $|\psi(0)\rangle$, finding the time-evolved state $|\psi(t)\rangle$
 is easy if you know the energy eigenstates $H|E_n\rangle = E_n|E_n\rangle$

Recall $\mathbb{1} = \sum_n |E_n\rangle \langle E_n|$

$$H = H\mathbb{1} = \sum_n H|E_n\rangle \langle E_n| = \sum_n E_n |E_n\rangle \langle E_n|$$

$$U = e^{-\frac{iHt}{\hbar}} = \sum_n e^{-\frac{iE_n t}{\hbar}} |E_n\rangle \langle E_n| = \sum_n e^{-\frac{iE_n t}{\hbar}} |E_n\rangle \langle E_n|$$

Then $|\psi(t)\rangle = U(t) |\psi(0)\rangle$

$$= \sum_n e^{-\frac{iE_n t}{\hbar}} |E_n\rangle \langle E_n | \psi(0) \rangle$$

but $\langle E_n | \psi(0) \rangle = c_n(0)$, coeff of $|\psi(0)\rangle$ in the energy basis

$$\Rightarrow |\psi(t)\rangle = \sum_n \underbrace{e^{-\frac{iE_n t}{\hbar}} c_n(0)}_{\equiv c_n(t)} |E_n\rangle$$

$\equiv c_n(t)$, coeff of $|\psi(t)\rangle$ in energy basis

Each eigenstate gets multiplied by a (different)
 complex phase $e^{-\frac{iE_n t}{\hbar}}$

only relative phases are physically significant

$$|\psi(t)\rangle = \underbrace{e^{-\frac{iE_1 t}{\hbar}}}_{\text{overall phase}} \sum_n c_n(0) \underbrace{e^{-\frac{i(E_n - E_1)t}{\hbar}}}_{\text{relative phases only depend on energy difference (as in CM)}} |E_n\rangle$$

overall phase
 physically irrelevant

(so overall zero of energy
 "irrelevant")