

(Recall)

A spin- j representation of angular momentum
is a vector space of dimension $2j+1$.

Call this space $(2j+1)$

It is spanned by basis states $|j, m\rangle$ where $m = -j, \dots, j$
(eigenstates of J^2 and J_z)

Most general state is linear combination
 $\sum c_m |j, m\rangle$

E.g. spin- $\frac{1}{2}$ rep, or $\mathbb{C}^2$ is spanned by $|\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle$
 $|\frac{1}{2}\rangle + |-\frac{1}{2}\rangle$

For brevity, we'll often omit j :

or even more briefly $|+\rangle + |-\rangle$

Recall (for post-Set) $J_+ |j, m\rangle = \hbar \sqrt{(j-m)(j+m+1)} |j, m+1\rangle$

$J_- |j, m\rangle = \hbar \sqrt{(j+m)(j-m+1)} |j, m-1\rangle$

↓
or wait
until turned in

Consider a system of two spins: s_1 and s_2 eg proton & electron in ground state of hydrogen

Basis states of the system are written

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle, \text{ or } |m_1\rangle \otimes |m_2\rangle, \sim |m_1; m_2\rangle$$

Product space is $(2j_1+1)(2j_2+1)$ dimensional (NB. $j_1 + j_2$ are implied)

eg two spin- $\frac{1}{2}$ particles, both spin up $\rightarrow |\frac{1}{2}; \frac{1}{2}\rangle$
 " " down $\rightarrow |-\frac{1}{2}; -\frac{1}{2}\rangle$
 also $|\frac{1}{2}; -\frac{1}{2}\rangle$
 $|-\frac{1}{2}; \frac{1}{2}\rangle$

This space, called $\underline{2} \otimes \underline{2}$ is 4-dimensional
 (tensor) product space

more generally, $(\underline{2j_1+1}) \otimes (\underline{2j_2+1})$ is $(2j_1+1)(2j_2+1)$ dimensional

The most general state of the system is a linear combination $\sum_{m_1, m_2} c_{m_1, m_2} |m_1; m_2\rangle$

Electron in hydrogen atom

Old quantum theory: described by 4 quantum numbers (Pauli, 1924)
 principal $n = 1, 2, 3, \dots \Rightarrow$ determines energy $E_n = \frac{-13.6 \text{ eV}}{n^2}$

orbital $l = 0, 1, \dots, n-1 \Rightarrow |\vec{L}| = l\hbar$

(magnetic) $m = -l, -l+1, \dots, l \Rightarrow L_z = m\hbar$

$s = \frac{1}{2} \Rightarrow |\vec{S}| = \frac{1}{2}\hbar$
 spin $m_s = \pm \frac{1}{2} \Rightarrow S_z = m_s \hbar$

Quantum mechanics

$|l, m\rangle$ describe states of orbital angular momentum

$|\frac{1}{2}, m_s\rangle$ describe state of spin of electron (intrinsic)

Therefore, apart from the principal quantum number
 states of an electron are described by the tensor products
 $|l, m\rangle \otimes |\frac{1}{2}, m_s\rangle$ which belong to $2l+1 \otimes 2$

(First time they're seen this)

n	$l=0$	$l=1$	$l=2$
$n=4$	<u>102</u> (4s)	<u>302</u> (4p)	<u>502</u> (4d)
$n=3$	<u>102</u> (3s)	<u>302</u> (3p)	<u>502</u> (3d)
$n=2$	<u>102</u> (2s)	<u>302</u> (2p)	
$n=1$	<u>102</u> (1s)		

Exclusion principle $\Rightarrow$ periodic table

An operator on the (tensor) product space is of the form $A \otimes B$ where A acts on 1st part & B on the 2nd

$$A \otimes B |m_1; m_2\rangle = A \otimes B |m_1\rangle \otimes |m_2\rangle \equiv A|m_1\rangle \otimes B|m_2\rangle$$

Eg. spin of 1st particle $\vec{J}_1$ is $\vec{J}_1 \otimes 1$
 "... 2nd particle $\vec{J}_2$ is $1 \otimes \vec{J}_2$

Eg. $J_{1z} |\frac{1}{2}; -\frac{1}{2}\rangle = \frac{1}{2}\hbar \cdot 1 |\frac{1}{2}; -\frac{1}{2}\rangle$
 $J_{2z} |\frac{1}{2}; -\frac{1}{2}\rangle = 1 \cdot (-\frac{1}{2}) |\frac{1}{2}; -\frac{1}{2}\rangle$

so $|m_1; m_2\rangle$ are eigenstates of $J_1^2, J_{1z}, J_2^2, J_{2z}$
 as we already know

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Product of operators

Note that $\vec{J}_1$ and $\vec{J}_2$ commute w one another $\Rightarrow$ compatible

$$\left(\begin{aligned} (\vec{J}_1 \otimes 1)(1 \otimes \vec{J}_2) &= (\vec{J}_1 \otimes \vec{J}_2) \\ (1 \otimes \vec{J}_2)(\vec{J}_1 \otimes 1) &= \vec{J}_1 \otimes \vec{J}_2 \end{aligned} \right) \checkmark$$

ie any component of $\vec{J}_1$ commutes
 w any component of $\vec{J}_2$

Total angular momentum

11-5

Classically, ^{total} spin of the system is just $\vec{J} = \vec{J}_1 + \vec{J}_2$

Qm, total spin is an operator $\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$

Act w/ $\vec{J}_z$ on the ~~the~~ product space basis $2 \otimes 2$

$$J_z \left| \frac{1}{2}; \frac{1}{2} \right\rangle = \frac{\hbar}{2} \left| \frac{1}{2}; \frac{1}{2} \right\rangle + \frac{\hbar}{2} \left| \frac{1}{2}; \frac{1}{2} \right\rangle = \hbar \left| \frac{1}{2}; \frac{1}{2} \right\rangle$$

$$J_z \left| -\frac{1}{2}; -\frac{1}{2} \right\rangle = -\hbar \left| -\frac{1}{2}; -\frac{1}{2} \right\rangle$$

$$J_z \left| \frac{1}{2}; -\frac{1}{2} \right\rangle = \frac{\hbar}{2} \left| \frac{1}{2}; \frac{1}{2} \right\rangle - \frac{\hbar}{2} \left| \frac{1}{2}; -\frac{1}{2} \right\rangle = 0$$

$$J_z \left| -\frac{1}{2}; \frac{1}{2} \right\rangle = 0$$

what rep. of total spin is this?

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \text{spin } 1 & \text{and} & \text{spin } 0 \\ \downarrow & & \downarrow \\ (\text{spin } \frac{1}{2}) \otimes (\text{spin } \frac{1}{2}) & = & (\text{spin } 1) \oplus (\text{spin } 0) \end{array}$$

$$2 \otimes 2 = 3 \oplus 1$$

(tensor) product space

direct sum

Tensor product space
of individual spins

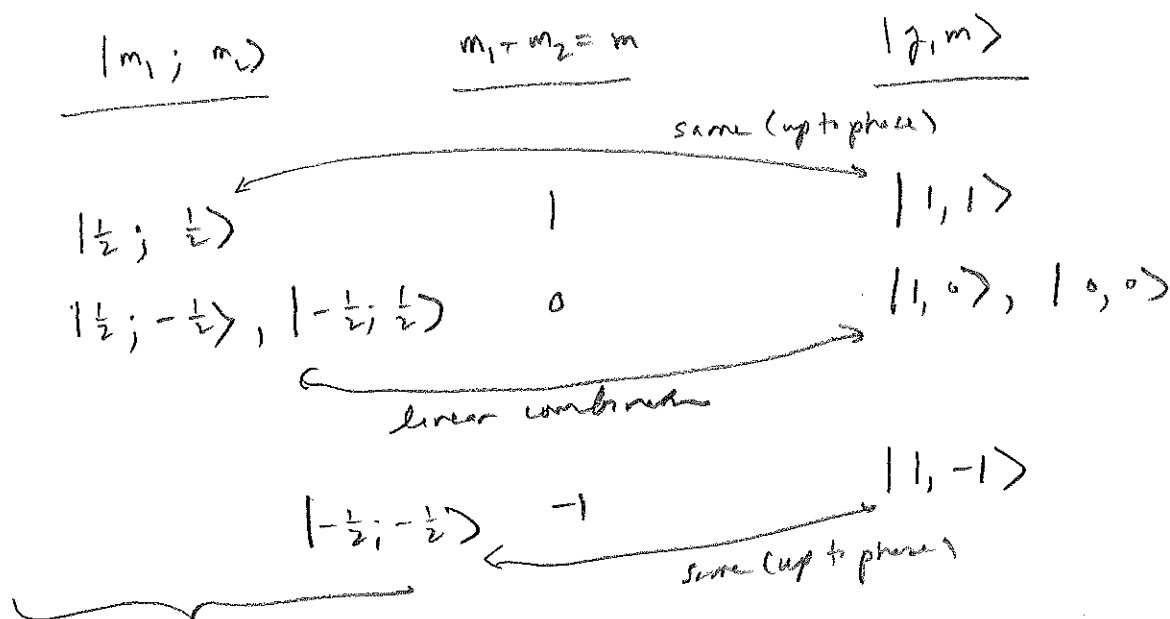
Direct sum space
of total spins of system

$$(\text{spin } \frac{1}{2}) \otimes (\text{spin } \frac{1}{2})$$

$$(\text{spin } 1) \oplus (\text{spin } 0)$$

$$\text{or } \underline{2} \oplus \underline{2}$$

$$\underline{3} \oplus \underline{1}$$



Eigenstates of $J_{1z} + J_{2z}$ ($+J_1^2 + J_2^2$)

Eigenstates of J_z and J^2

Also eigenstates of J_z because $J_z = J_{1z} + J_{2z}$

Not (necessarily) eigenstates of J^2
(because $[J^2, J_{1z}] \neq 0$
 $[J^2, J_{2z}] \neq 0$)

Not (necessarily) eigenstates
of J_{1z} and J_{2z}

11-7

The only states on each side w/ $m = m_1 + m_2 = 1$ are

$$|1, 1\rangle \quad \text{and} \quad |\frac{1}{2}; \frac{1}{2}\rangle$$

so these must be proportional $|1, 1\rangle = e^{i\phi} |\frac{1}{2}; \frac{1}{2}\rangle$

↓
choose $e^{i\phi} = 1$

$$\boxed{|1, 1\rangle = |\frac{1}{2}; \frac{1}{2}\rangle}$$

Act on both sides w/ J_-

$$J_- |1, 1\rangle = J_- |\frac{1}{2}; \frac{1}{2}\rangle$$

$$= (\hat{J}_{1-} \mathbb{1} + \mathbb{1} \otimes \hat{J}_{2-}) |\frac{1}{2}; \frac{1}{2}\rangle$$

$$\hbar\sqrt{2} |1, 0\rangle = \hbar |-\frac{1}{2}; \frac{1}{2}\rangle + \hbar |\frac{1}{2}; -\frac{1}{2}\rangle$$

$$\boxed{|1, 0\rangle = \frac{1}{\sqrt{2}} |-\frac{1}{2}; \frac{1}{2}\rangle + \frac{1}{\sqrt{2}} |\frac{1}{2}; -\frac{1}{2}\rangle}$$

Act again w/ J_-

$$J_- |1, 0\rangle = (\hat{J}_{1-} \mathbb{1} + \mathbb{1} \otimes \hat{J}_{2-}) \frac{1}{\sqrt{2}} (|-\frac{1}{2}; \frac{1}{2}\rangle + |\frac{1}{2}; -\frac{1}{2}\rangle)$$

$$\hbar\sqrt{2} |1, -1\rangle = \frac{1}{\sqrt{2}} (0 + \hbar |-\frac{1}{2}; -\frac{1}{2}\rangle + \hbar |-\frac{1}{2}; -\frac{1}{2}\rangle + 0)$$

$$\boxed{|1, -1\rangle = |-\frac{1}{2}; -\frac{1}{2}\rangle}$$

We could now guess

$$|0, 0\rangle = \frac{1}{\sqrt{2}} |-\frac{1}{2}; \frac{1}{2}\rangle - \frac{1}{\sqrt{2}} |\frac{1}{2}; -\frac{1}{2}\rangle$$

using $\langle 0, 0 | 1, 0 \rangle = 0$ but do it another way

What about $|0, 0\rangle$?

We know $J_+ |0, 0\rangle = 0$

$$\text{Let } |0, 0\rangle = \alpha \left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \beta \left| -\frac{1}{2}; \frac{1}{2} \right\rangle$$

$$J_+ |0, 0\rangle = (J_{1+} \otimes 1 + 1 \otimes J_{2+}) (\alpha \left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \beta \left| -\frac{1}{2}; \frac{1}{2} \right\rangle)$$

$$= 0 + \alpha \hbar \left| \frac{1}{2}; \frac{1}{2} \right\rangle + \beta \hbar \left| \frac{1}{2}; \frac{1}{2} \right\rangle + 0$$

$$= (\alpha + \beta) \hbar \left| \frac{1}{2}; \frac{1}{2} \right\rangle$$

$$\Rightarrow \beta = -\alpha$$

$$\text{Normalized: } \alpha = \frac{1}{\sqrt{2}}$$

$$\text{Singlet } |0, 0\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}; -\frac{1}{2} \right\rangle - \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right]$$

Prob: find CG coeffs

$$\hbar \hat{L} \otimes \hat{L} = \hat{L} \otimes \hat{L}$$

Problem

Calc $|\frac{3}{2}, m\rangle, |\frac{1}{2}, m\rangle$

in $\text{Spin } 1 \otimes \text{Spin } \frac{1}{2}$

$$J_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

$$J_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$|\frac{3}{2}, \frac{3}{2}\rangle = |1; \frac{1}{2}\rangle$$

$$\sqrt{3} |\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{2} |0; \frac{1}{2}\rangle + |1; -\frac{1}{2}\rangle$$

$$2\sqrt{3} |\frac{3}{2}, -\frac{1}{2}\rangle = 2|-1; \frac{1}{2}\rangle + \sqrt{2} |0; -\frac{1}{2}\rangle + \sqrt{2} |0; -\frac{1}{2}\rangle$$

$$6 |\frac{3}{2}, -\frac{3}{2}\rangle = 2|-1; -\frac{1}{2}\rangle + 2|-1; -\frac{1}{2}\rangle + 2|-1; -\frac{1}{2}\rangle$$

$$\left\{ \begin{aligned} |\frac{3}{2}, \frac{3}{2}\rangle &= |1; \frac{1}{2}\rangle \\ |\frac{3}{2}, \frac{1}{2}\rangle &= \sqrt{\frac{2}{3}} |0; \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |1; -\frac{1}{2}\rangle \\ |\frac{3}{2}, -\frac{1}{2}\rangle &= \sqrt{\frac{2}{3}} |0; -\frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |-1; \frac{1}{2}\rangle \\ |\frac{3}{2}, -\frac{3}{2}\rangle &= |-1; -\frac{1}{2}\rangle \end{aligned} \right.$$

$$|\frac{1}{2}, \frac{1}{2}\rangle = \alpha |0; +\frac{1}{2}\rangle + \beta |1; -\frac{1}{2}\rangle$$

$$J_+ |\frac{1}{2}, \frac{1}{2}\rangle = \alpha\sqrt{2} |1; +\frac{1}{2}\rangle + \beta |1; \frac{1}{2}\rangle = 0 \Rightarrow \beta = -\sqrt{2}\alpha$$

$$\Rightarrow |\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}} |0; \frac{1}{2}\rangle - \sqrt{\frac{2}{3}} |1; -\frac{1}{2}\rangle$$

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}} |-1; \frac{1}{2}\rangle + \sqrt{\frac{1}{3}} |0; -\frac{1}{2}\rangle - \frac{2}{\sqrt{3}} |0; -\frac{1}{2}\rangle$$

$$= \sqrt{\frac{2}{3}} |-1; \frac{1}{2}\rangle - \frac{1}{\sqrt{3}} |0; -\frac{1}{2}\rangle$$

$$\left\{ \begin{aligned} |\frac{1}{2}, \frac{1}{2}\rangle &= \sqrt{\frac{1}{3}} |0; +\frac{1}{2}\rangle - \sqrt{\frac{2}{3}} |1; -\frac{1}{2}\rangle \\ |\frac{1}{2}, -\frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}} |0; -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}} |-1; \frac{1}{2}\rangle \end{aligned} \right.$$

Adding angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$

Suppose we have two particles with angular momentum $\vec{J}_1$ and $\vec{J}_2$. The state space of the system is the product space of the individual state spaces.

A state of the system of two particles belongs to the product space.

$$(Spin J_1) \otimes (Spin J_2)$$

$$\sim (2J_1+1) \otimes (2J_2+1)$$

spanned by basis states

$$|J_1, m_1; J_2, m_2\rangle \text{ or } |m_1, m_2\rangle \text{ for short}$$

$$m_1 = -J_1, \dots, J_1$$

$$m_2 = -J_2, \dots, J_2$$

Dimension of space
 $= (2J_1+1)(2J_2+1)$

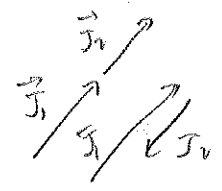
one can measure the total angular momentum of a state in the space and obtain various results J ranging from J_1+J_2 to $|J_1-J_2|$ by integer steps.

The space is therefore also a direct sum of representations of total angular momentum

$$J = J_1 + J_2$$

$$\oplus (Spin J)$$

$$J = |J_1 - J_2|$$



Spanned by states $|J, m\rangle$

$$J = |J_1 - J_2|, \dots, J_1 + J_2$$

$$m = -J, \dots, J$$

Example: $j_1=1, j_2=2$

$$(sp=1) \otimes (sp=2) = (sp=3) + (sp=2) + (sp=1)$$

$$\underline{3} \otimes \underline{2} = \underline{5} \oplus \underline{3} \oplus \underline{1}$$

Product space basis states

$$|m_1; m_2\rangle$$

$$|1; 2\rangle$$

$$|1; 1\rangle \quad |0; 2\rangle$$

$$|1; 0\rangle \quad |0; 1\rangle \quad |-1; 2\rangle$$

$$|1; -1\rangle \quad |0; 0\rangle \quad |-1; 1\rangle$$

$$|1; -2\rangle \quad |0; -1\rangle \quad |-1; 0\rangle$$

$$|0; -2\rangle \quad |-1; -1\rangle$$

$$|-1; -2\rangle$$

expansion of $J_z + J_{z2}$

$$J_z = J_{z1} \otimes 1_2 + 1_1 \otimes J_{z2}$$

$$m = m_1 + m_2$$

$$3$$

$$2$$

$$1$$

$$0$$

$$-1$$

$$-2$$

$$-3$$

Direct sum basis states

$$|j, m\rangle$$

$$|3, 3\rangle$$

$$|3, 2\rangle \quad |2, 2\rangle$$

$$|3, 1\rangle \quad |2, 1\rangle \quad |1, 1\rangle$$

$$|3, 0\rangle \quad |2, 0\rangle \quad |1, 0\rangle$$

$$|3, -1\rangle \quad |2, -1\rangle \quad |1, -1\rangle$$

$$|3, -2\rangle \quad |2, -2\rangle$$

$$|3, -3\rangle$$

expansion of J^2, J_z

Then states in any row on the right are linear combinations of states in same row on the left & vice versa

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ \text{where } m_1 + m_2 = m}} C_{m_1 m_2}^{j_1 j_2 j} |m_1; m_2\rangle$$

Clebsch-Gordan coefficients

$$eg \quad |3, 1\rangle = C_{10}^{123} |1; 0\rangle + C_{01}^{123} |0; 1\rangle + C_{-1, 2}^{1, 1, 3} |-1; 2\rangle$$

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$$J^2 = J_1^2 + 2\vec{J}_1 \cdot \vec{J}_2 + J_2^2 \quad \left. \vphantom{J^2 = J_1^2 + 2\vec{J}_1 \cdot \vec{J}_2 + J_2^2} \right\} \text{classically}$$

11-11

$$\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$$

$$J_i = J_{1i} \otimes 1 + 1 \otimes J_{2i} \quad \text{where } i = x, y, z$$

$$J^2 = \sum_i J_i J_i = \sum_i (J_{1i} \otimes 1 + 1 \otimes J_{2i})(J_{1i} \otimes 1 + 1 \otimes J_{2i})$$

$$= \sum_i (J_{1i} J_{1i} \otimes 1 + 2 J_{1i} \otimes J_{2i} + 1 \otimes J_{2i} J_{2i})$$

$$= J_1^2 \otimes 1 + 2 (J_{1z} \otimes J_{2z} + J_{1x} \otimes J_{2x} + J_{1y} \otimes J_{2y}) + 1 \otimes J_2^2$$

$$\text{using } J_{1x} = \frac{1}{2}(J_{1+} + J_{1-})$$

$$J_{1y} = \frac{1}{2i}(J_{1+} - J_{1-})$$

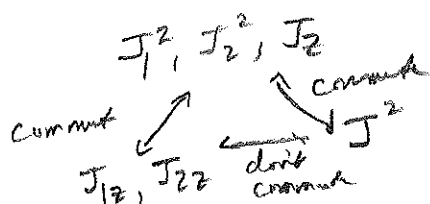
etc

$$J^2 = J_1^2 \otimes 1 + 1 \otimes J_2^2 + 2 J_{1z} \otimes J_{2z} + J_{1+} \otimes J_{2-} + J_{1-} \otimes J_{2+}$$

Note $[J^2, J_{1z} \otimes 1] \neq 0$

$[J^2, 1 \otimes J_{2z}] \neq 0$

As eigenstates of J_{1z} & J_{2z} are not (necessarily) eigenstates of J^2



Consider highest state of product rep

$$J^2 |j_1, j_2\rangle = \underbrace{[j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 + 2j_1j_2\hbar^2 + 0 + 0]}_{(j_1+j_2)(j_1+j_2+1)\hbar^2} |j_1, j_2\rangle$$

$$\text{so } |j_1, j_2\rangle = |j_1+j_2, j_1+j_2\rangle$$

$$J_- |1,1\rangle = \sqrt{2} |1,0\rangle$$

$$J_- |1,0\rangle = \sqrt{2} |1,-1\rangle$$

$$J^2 = J_1^2 + J_2^2 + 2 J_{1z} J_{2z} + J_{1+} J_{2-} + J_{1-} J_{2+}$$

$$J^2 |1, \frac{1}{2}\rangle = 2 + \frac{3}{4} + 2(1)(\frac{1}{2}) = \frac{15}{4} = \frac{3}{2}(\frac{3}{2} + 1) \quad \checkmark$$

$$J^2 |\frac{3}{2}, \frac{1}{2}\rangle = (\sqrt{\frac{2}{3}} |0, \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |1, -\frac{1}{2}\rangle)$$

$$= (2 + \frac{3}{4}) |\frac{3}{2}, \frac{1}{2}\rangle + \sqrt{\frac{2}{3}} (\sqrt{2} |1, -\frac{1}{2}\rangle) + \frac{1}{\sqrt{3}} (2(1)(-\frac{1}{2}) |1, -\frac{1}{2}\rangle + \sqrt{2} |0, \frac{1}{2}\rangle)$$

$$\frac{1}{\sqrt{3}} |1, -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}} |0, \frac{1}{2}\rangle$$

$$|\frac{3}{2}, \frac{1}{2}\rangle$$

$$= \frac{15}{4} |\frac{3}{2}, \frac{1}{2}\rangle \quad \checkmark$$

$$J^2 |\frac{1}{2}, \frac{1}{2}\rangle = J^2 (\sqrt{\frac{1}{3}} |0, \frac{1}{2}\rangle - \sqrt{\frac{2}{3}} |1, -\frac{1}{2}\rangle)$$

$$= (2 + \frac{3}{4}) |\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{1}{3}} (\sqrt{2} |1, -\frac{1}{2}\rangle) - \sqrt{\frac{2}{3}} (2(1)(-\frac{1}{2}) |1, -\frac{1}{2}\rangle + \sqrt{2} |0, \frac{1}{2}\rangle)$$

$$2\sqrt{\frac{2}{3}} |1, -\frac{1}{2}\rangle - 2\frac{1}{\sqrt{3}} |0, \frac{1}{2}\rangle$$

$$-2 |\frac{1}{2}, \frac{1}{2}\rangle$$

$$= \frac{3}{4} |\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{2}(1 + \frac{1}{2}) |\frac{1}{2}, \frac{1}{2}\rangle \quad \checkmark$$

midterm question

$$J^2 = J_1^2 \otimes 1 + 1 \otimes J_2^2 + 2J_{1z} \otimes J_{2z} + J_{1+} \otimes J_{2-} + J_{1-} \otimes J_{2+}$$

$$\begin{aligned} J^2 \left| \frac{1}{2}; \frac{1}{2} \right\rangle &= \left[\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) + 0 + 0 \right] \left| \frac{1}{2}; \frac{1}{2} \right\rangle \\ &= 2\hbar^2 \left| \frac{1}{2}; \frac{1}{2} \right\rangle \end{aligned}$$

Thus $|1, 1\rangle = \left| \frac{1}{2}; \frac{1}{2} \right\rangle$

$$\begin{aligned} J^2 \left| \frac{1}{2}; -\frac{1}{2} \right\rangle &= \underbrace{\left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right) \right)}_{\hbar^2} \left| \frac{1}{2}; -\frac{1}{2} \right\rangle + 0 + \hbar^2 \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \\ &= \hbar^2 \left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \hbar^2 \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \end{aligned}$$

Similarly

$$J^2 \left| -\frac{1}{2}; \frac{1}{2} \right\rangle = \hbar^2 \left| -\frac{1}{2}; \frac{1}{2} \right\rangle + \hbar^2 \left| \frac{1}{2}; -\frac{1}{2} \right\rangle$$

$$\therefore J^2 \left(\left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right) = 2\hbar^2 \left(\left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right)$$

Thus $|1, 0\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}; -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right)$

$$J^2 \left(\left| \frac{1}{2}; -\frac{1}{2} \right\rangle - \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right) = 0$$

Thus $|0, 0\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}; -\frac{1}{2} \right\rangle - \left| -\frac{1}{2}; \frac{1}{2} \right\rangle \right)$