Derivation of general representation of angular momentum algebra

Let $\vec{J} = (J_x, J_y, J_z)$ be a set of hermitian operators obeying

$$[J_x, J_y] = i\hbar J_z$$
$$[J_y, J_z] = i\hbar J_x \qquad (*)$$
$$[J_z, J_x] = i\hbar J_y$$

Recall $J^2 = J_x^2 + J_y^2 + J_z^2$

[earlier we saw that for spin $\frac{1}{2}$, S^2 and S_z are compatible because $|\pm\rangle$ are eigenstates of both.]

Using $(*)$, you showed $[J_z, J^2] = 0$

$\therefore$ J^2 and J_z are compatible and have a complete set of mutual eigenstates $|\alpha, m\rangle$

$$J^2 |\alpha, m\rangle = \alpha \hbar^2 |\alpha, m\rangle$$
$$J_z |\alpha, m\rangle = m\hbar |\alpha, m\rangle$$

[NB we do not assume α, m are integers]

Let $|\alpha, m\rangle$ form an orthonormal basis

Recall $J_\pm = J_x \pm i J_y$

You showed $[J_z, J_\pm] = \pm \hbar J_\pm$

so $$J_z \left(J_\pm |\alpha, m\rangle\right) = (m \pm 1)\hbar \left(J_\pm |\alpha, m\rangle\right)$$

ie $J_\pm$ raises/lowers the J_z eigenvalue by 1

You showed
$$[J^2, J_x] = 0$$
$$[J^2, J_y] = 0$$
$$\Rightarrow [J^2, J_\pm] = 0$$

$$J^2 \left(J_\pm |\alpha, m\rangle\right) = \alpha \hbar^2 \left(J_\pm |\alpha, m\rangle\right)$$

ie $J_\pm$ does not change the J^2 eigenvalue

$$J_\pm |\alpha, m\rangle \sim |\alpha, m \pm 1\rangle$$

To prove: $\alpha \geq m^2$

$$J^2 = J_z^{\,2} + J_y^{\,2} + J_x^{\,2}$$

$$\langle \alpha, m | J^2 | \alpha, m \rangle = \langle \alpha, m | J_z^2 | \alpha, m \rangle + \langle \alpha, m | J_y^2 | \alpha, m \rangle + \langle \alpha, m | J_x^2 | \alpha, m \rangle$$

$$\alpha \hbar^2 = m^2 \hbar^2 + \quad ? \quad + \quad ??$$

Lemma: for any hermitian operator A, $\langle \psi | A^2 | \psi \rangle \geq 0$

$\langle \psi | A^2 | \psi \rangle = \langle \psi | A^{\dagger} A | \psi \rangle = \langle \chi | \chi \rangle \geq 0$

J_x and J_y are hermitian so ? and ?? are ≥ 0

$\Rightarrow \alpha \geq m^2$

Hence J_+, J_- cannot operate indefinitely

$\exists$ a maximal value of m, call it j $\ni$ $J_+ | \alpha, j \rangle = 0$

and a minimum value, call it $\bar{j}$ $\ni$ $J_- | \alpha, \bar{j} \rangle = 0$

$$\left. \begin{matrix} J_+ \\ J_- \quad |\alpha, j\rangle \\ \quad\; |\alpha, j-1\rangle \\ \vdots \\ J_- \quad |\alpha, \bar{j}\rangle \\ J_- \end{matrix} \right\}$$

This finite set of states is a "representation of angular momentum algebra"

\# of states $= j - \bar{j} + 1$

(m's not necessarily integers)

We previously showed $J^2 = J_z^2 + \hbar J_z + J_- J_+$

Act on top state

$$J^2|\alpha, j\rangle = (J_z^2 + \hbar J_z + J_- J_+)|\alpha, j\rangle$$

$$\alpha\hbar^2|\alpha, j\rangle = ((j\hbar)^2 + \hbar(j\hbar) + 0)|\alpha, j\rangle$$

$$\Rightarrow \alpha = j(j+1)$$

We also showed $J^2 = J_z^2 - \hbar J_z + J_+ J_-$

Act on lowest state

$$\alpha\hbar^2|\alpha, \bar{j}\rangle = ((\bar{j}\hbar)^2 - \hbar(\bar{j}\hbar) + 0)|\alpha, \bar{j}\rangle$$

$$\alpha = \bar{j}(\bar{j}-1)$$

$$j(j+1) = \bar{j}(\bar{j}-1)$$

$$j^2 - \bar{j}^2 + j + \bar{j} = 0$$

$$(j+\bar{j})(j-\bar{j})$$

$$(j+\bar{j})(j-\bar{j}+1) = 0$$

$\bar{j} = \begin{cases} -j \\ \cancel{j+1} \end{cases}$ because $\bar{j} < j$

$$\left.\begin{matrix} |\alpha, j\rangle \\ \vdots \\ |\alpha, \bar{j}\rangle \end{matrix}\right\}$$

\# j states $= j - \bar{j} + 1 = 2j+1 \in \mathbb{Z}$

j can be either integer (orbital ang momentum / or spin j boson)

or

half integer (spin j fermion)

"spin j representation"

or $2j+1$

~~Most general [illegible] of the rep~~

Change in notation:

instead of $|\alpha, m\rangle$ w/ $\alpha = j(j+1)$

we write $|j, m\rangle$ where $j = m_{max}$

$$J^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle$$
$$J_z |j, m\rangle = m\hbar |j, m\rangle$$

Assuming that $|j, m\rangle$ is an orthonormal basis,

$$J_+ |j, m\rangle = c_+(j, m) |j, m+1\rangle$$
$$J_- |j, m\rangle = c_-(j, m) |j, m-1\rangle$$

[Problem: calculate $c_+(j, m)$ and $c_-(j, m)$]

$$\left[\begin{array}{l} \text{Ans: } c_+ = \sqrt{j(j+1) - m(m+1)}\ \hbar = \sqrt{(j-m)(j+m+1)}\,\hbar \\ \qquad\ c_- = \sqrt{j(j+1) - m(m-1)}\ \hbar = \sqrt{(j+m)(j-m+1)}\,\hbar \end{array}\right]$$

Most general state of spin j is $|\psi\rangle = \sum_{m=-j}^{j} a_m |j, m\rangle$

has well-defined J^2 but not J_z