Transformation from A-basis to B-basis

$$A|a_n\rangle = a_n|a_n\rangle \quad \Rightarrow \quad \Sigma |a_n\rangle\langle a_n| = 1$$

$$B|b_n\rangle = b_n|b_n\rangle \quad \Rightarrow \quad \Sigma |b_n\rangle\langle b_n| = 1$$

(Sakurai, p. 36)

Ket $|\psi\rangle$ is represented in B-basis as column vector w/ components

$$\psi_m' = \langle b_m|\psi\rangle = \sum_k \langle b_m|a_k\rangle\langle a_k|\psi\rangle = \Sigma \langle b_m|a_k\rangle \psi_k$$

components in A-basis

Operator C is represented in B-basis as matrix element

$$C'_{mn} = \langle b_m|C|b_n\rangle = \sum_{k,l} \langle b_m|a_k\rangle\langle a_k|C|a_l\rangle\langle a_l|b_n\rangle$$

$$= \Sigma \langle b_m|a_k\rangle C_{kl} \langle a_l|b_n\rangle$$

matrix element in A-basis

Define transformation operator from A-basis to B-basis as

$$\boxed{\hat{U} = \sum_n |b_n\rangle\langle a_n|}$$

$\hat{U}$ converts $|a_\ell\rangle$ into $|b_\ell\rangle$

$$\hat{U}|a_\ell\rangle = \sum_n |b_n\rangle \underbrace{\langle a_n|a_\ell\rangle}_{\delta_{n\ell}} = |b_\ell\rangle$$

Hermitian conjugate operator

$$\boxed{\hat{U}^\dagger = \sum_n |a_n\rangle\langle b_n|}$$

$\hat{U}$ is a unitary operator, which means $\hat{U}^\dagger\hat{U} = \mathbb{1}$

check: $\hat{U}^\dagger\hat{U} = \left(\sum_n |a_n\rangle\langle b_n|\right)\left(\sum_\ell |b_\ell\rangle\langle a_\ell|\right)$

$$= \sum_{n\ell} |a_n\rangle \underbrace{\langle b_n|b_\ell\rangle}_{\delta_{n\ell}} \langle a_\ell| = \sum |a_n\rangle\langle a_n| = \mathbb{1}$$

$\hat{U}$ corresponds to a matrix $U$ w/ matrix elements

$$U_{\ell n} = \langle a_\ell|\hat{U}|a_n\rangle = \sum_m \langle a_\ell|b_m\rangle\langle a_m|a_n\rangle = \langle a_\ell|b_n\rangle$$

$$U_{\ell n} = \langle b_\ell|\hat{U}|b_n\rangle = \sum_m \langle b_\ell|b_m\rangle\langle a_m|b_n\rangle = \langle a_\ell|b_n\rangle$$

same in both bases!

$\hat{U}^\dagger$ has matrix elements

$$\underbrace{(U^\dagger)_{mk}} = \langle a_m|\hat{U}^\dagger|a_k\rangle = \langle b_m|a_k\rangle$$

$\hookrightarrow$ should be $(U^{T*})_{mk} = U^*_{km} = \langle a_k|b_m\rangle^* = \langle b_m|a_k\rangle$ ✓

Recall

$$\psi_m' = \sum_k \langle b_m | a_k \rangle \psi_k = \sum (U^\dagger)_{mk} \psi_k = (U^\dagger \psi)_m$$

$$C_{mn}' = \sum \langle b_m | a_k \rangle C_{k\ell} \langle a_\ell | b_n \rangle$$

$$= \sum (U^\dagger)_{mk} C_{k\ell} U_{\ell n}$$

$$= (U^\dagger C U)_{mn}$$

As matrix equation

$$\boxed{\psi' = U^\dagger \psi}$$

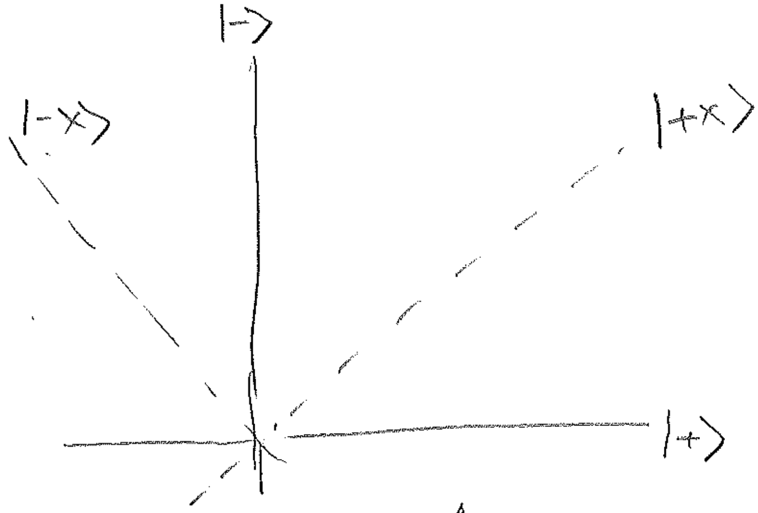
$$\boxed{C' = U^\dagger C U}$$ (similarity transformation)

Since $U^\dagger U = 1 \Rightarrow U^\dagger = U^{-1}$

$$\psi' = U^{-1} \psi$$

$$U\psi' = \psi$$

$$\boxed{\psi = U \psi'}$$




$$|+x\rangle = \frac{1}{\sqrt{2}}\left( |+\rangle + |-\rangle \right)$$
$$|-x\rangle = \frac{1}{\sqrt{2}}\left( -|+\rangle + |-\rangle \right)$$

picture only possible because coeffs are real

Observe: $|\pm\rangle$ are orthogonal

$|\pm x\rangle$ are orthogonal

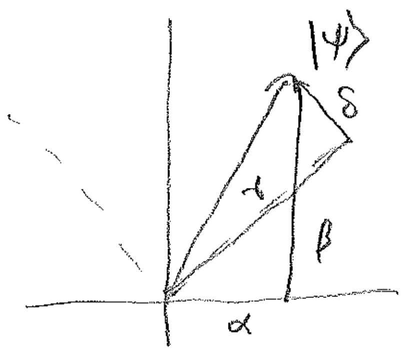
$|\pm\rangle$ and $|\pm x\rangle$ are related by a $45°$ rotation in state space

(more generally, by a <u>unitary</u> transformation)

"complex rotation"

Let $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \Rightarrow$ vector in $S_z$-basis $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$|\psi\rangle = \gamma|+x\rangle + \delta|-x\rangle \Rightarrow$ vect in $S_x$-basis $\psi' = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$




Can use geometry to relate $\psi$ & $\psi'$

namely $\gamma = \frac{1}{\sqrt{2}}(\alpha+\beta)$

$\delta = \frac{1}{\sqrt{2}}(-\alpha+\beta)$

but let's use unitary transformation

$U$ transforms from $S_z$-basis to $S_x$-basis

← [can just compute this using ket-bras]

$$= |+x\rangle\langle +| + |-x\rangle\langle -|$$

$U$ has matrix elements $\begin{pmatrix} \langle +|+x\rangle & \langle +|-x\rangle \\ \langle -|+x\rangle & \langle -|-x\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$U^\dagger = (U^{T*}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

check unitarity: $U^\dagger U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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$$\psi' = U^\dagger \psi$$

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha+\beta \\ -\alpha+\beta \end{pmatrix} \quad ✓$$

$$\psi = U\psi'$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \gamma-\delta \\ \gamma+\delta \end{pmatrix}$$

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$S_x$ in $S_z$-basis is $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

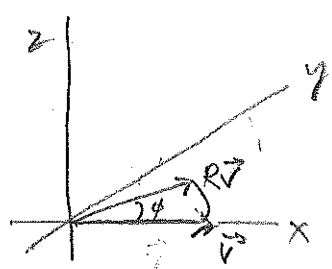
What is $S_x$ in $S_x$-basis?

$$S_x' = U^\dagger S_x U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{\hbar}{2} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ✓$$

## Rotations

implemented on vectors in physical space by an orthogonal $3\times 3$ rotation matrix $R$

orthogonal matrix means $R^T R = \mathbb{1}$




Consider rotation through $\phi$ counterclockwise about $\hat z$-axis

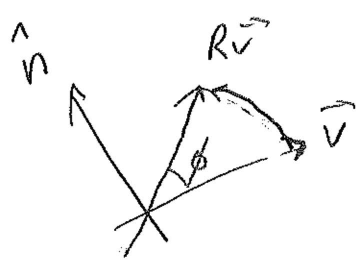
$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[NB opposite Phys 2140 because active not passive] $\vec v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow R\vec v = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$

careful w/ signs!

More generally, consider rotation through angle $\phi$ about some axis $\vec n$, where $\vec n \cdot \vec n = 1$ [don't use $\hat n$: operator]

[thumb along $\vec n$, fingers of right hand curl in direction of rotation]



Rotations

implemented on kets in state (Hilbert) space by a unitary matrix $U$

unitary matrix means $U^\dagger U = \mathbb{1}$

Claim: a rotation through $\phi$ about $\vec{n}$ implemented via

$$U = e^{-\frac{i\phi}{\hbar}\vec{n}\cdot\vec{J}}$$

where $\vec{J}$ = angular momentum operator

we say "angular momentum is a generator of rotations"

classical mechanics: Noether's theorem says
invariance of laws of physics under rotations implies
existence of a conserved quantity, viz. angular momentum

QM: ang. mom operator generates rotations (via $U$)

Consider a system consisting of spin-$\frac{1}{2}$ particle

$\vec{J} = \vec{S}$ (spin operator)

$\vec{S}$ of spin-$\frac{1}{2}$ particle is represented (in $S_z$ basis) by $\frac{\hbar}{2}\vec{\sigma}$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\vec{n}\cdot\vec{\sigma} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$, $\vec{n}\cdot\vec{n} = 1$

$$U = e^{-\frac{i\phi}{\hbar}\vec{n}\cdot\vec{S}} = e^{-\frac{i\phi}{2}\vec{n}\cdot\vec{\sigma}}$$

$$= \mathbb{1} - \frac{i\phi}{2}(\vec{n}\cdot\vec{\sigma}) + \frac{1}{2}\left(-\frac{i\phi}{2}\right)^2(\vec{n}\cdot\vec{\sigma})^2 + \frac{1}{3!}\left(-\frac{i\phi}{2}\right)^3(\vec{n}\cdot\vec{\sigma})^3 + \dots$$

$\underline{\underline{\text{can show:}}}$ $(\vec{n}\cdot\vec{\sigma})^2 = \mathbb{1}$, $(\vec{n}\cdot\vec{\sigma})^3 = \vec{n}\cdot\vec{\sigma}$, etc so

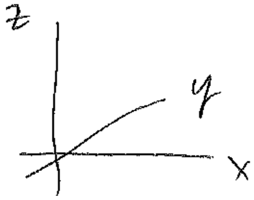
$$U = \left[1 - \frac{1}{2}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 + \dots\right]\mathbb{1} - i\left[\frac{\phi}{2} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \dots\right]\vec{n}\cdot\vec{\sigma}$$

$$= \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)(\vec{n}\cdot\vec{\sigma})$$

$$= \begin{bmatrix} \cos\frac{\phi}{2} - in_z\sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + in_z\sin\frac{\phi}{2} \end{bmatrix}$$

$$U^\dagger = \cos\left(\frac{\phi}{2}\right)\mathbb{1} + i\sin\left(\frac{\phi}{2}\right)(\vec{n}\cdot\vec{\sigma})$$ because $\vec{\sigma}^\dagger = \vec{\sigma}$

check: $U^\dagger U = \cos^2\left(\frac{\phi}{2}\right)\mathbb{1} + \sin^2\left(\frac{\phi}{2}\right)\underbrace{(\vec{n}\cdot\vec{\sigma})^2}_{\mathbb{1}} = \mathbb{1}$ ✓




If we want $\hat{z}$-axis $\to$ $\hat{x}$-axis

rotate about $\hat{n} = \hat{y}$ by $\phi = \frac{\pi}{2}$

This is implemented via

$$U = \underbrace{\cos\left(\frac{\pi}{4}\right)}_{1/\sqrt{2}} \mathbb{1} - i \underbrace{\sin\left(\frac{\pi}{4}\right)}_{1/\sqrt{2}} \sigma_y \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

agrees with what we found earlier

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Note: consider rotation about any axis by $2\pi$ radians

$$U = (\underbrace{\cos \pi}_{-1}) \mathbb{1} - i \underbrace{\sin(\pi)}_{0} \ \hat{\omega} \cdot \vec{\sigma}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi' = U^{+} \psi = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

↑ changes sign

(unobservable unless do interference experiment) → discuss later after electron precession

[Sakurai]