

Matrix representations in the $\hat{A}$ -basis

Given an orthonormal basis

(e.g. d nondegenerate eigenstates $|a_n\rangle$ of Hermitian operator A)

any operator B can be represented as a $d \times d$ matrix whose matrix elements in the A -basis are

$$B_{mn} = \langle a_m | B | a_n \rangle$$

For example, A in the A -basis is represented by

$$A_{mn} = \langle a_m | A | a_n \rangle = a_m \langle a_m | a_n \rangle = a_m \delta_{mn} \quad \text{use orthonormality}$$

$$A \rightarrow \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_d \end{pmatrix}, \text{ a diagonal matrix}$$

The product of 2 operators is represented by the matrix product

$$\begin{aligned} \langle a_m | BC | a_n \rangle &= \langle a_m | B \mathbb{1} C | a_n \rangle \stackrel{\text{use completeness}}{=} \sum_l \langle a_m | B | a_l \rangle \langle a_l | C | a_n \rangle \\ &= \sum_l B_{ml} C_{ln} = (BC)_{mn} \end{aligned}$$

Reconstruct operator from matrix elements

$$B_{mn} \rightarrow B = \sum_{m,n=1}^d |a_m\rangle B_{mn} \langle a_n|$$

in the A-basis
A ket $|\psi\rangle$ is represented as a d -dim column vector with entries $\psi_n = \langle a_n | \psi \rangle$

$$|\psi\rangle \rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix} = \begin{pmatrix} \langle a_1 | \psi \rangle \\ \langle a_2 | \psi \rangle \\ \vdots \\ \langle a_d | \psi \rangle \end{pmatrix}$$

Then basis kets are

$$|a_n\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow n^{\text{th}} \text{ entry}$$

Reconstruct ket from components $\psi_n \rightarrow |\psi\rangle = \sum_{n=1}^d \psi_n |a_n\rangle$

A bra $\langle\psi|$ is represented in the A-basis as a d -dim row vector with entries $\psi_n^* = \langle\psi|a_n\rangle = \langle a_n|\psi\rangle^*$

$$\langle\psi| \rightarrow (\psi_1^*, \psi_2^*, \dots, \psi_d^*) = (\langle\psi|a_1\rangle, \dots, \langle\psi|a_d\rangle)$$

$$\langle a_n| \rightarrow (0, 0, \dots, 1, \dots, 0) \leftarrow n^{\text{th}} \text{ entry}$$

Reconstruct

$$\langle\psi| = \sum \psi_n^* \langle a_n|$$

Inner product

$$\langle \phi | \psi \rangle = \sum_{n=1}^d \langle \phi | a_n \rangle \langle a_n | \psi \rangle = \sum \langle a_n | \phi \rangle^* \langle a_n | \psi \rangle = \sum \phi_n^* \psi_n$$

is just the product of row + column vectors

$$(\phi_1^*, \dots, \phi_d^*) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

Eigenvalue equation

(possibly unit)

$$B | \psi \rangle = \lambda | \psi \rangle$$

$$\sum_{m=1}^d \langle a_n | B | a_m \rangle \langle a_m | \psi \rangle = \lambda \langle a_n | \psi \rangle$$

$$\sum_m B_{nm} \psi_m = \lambda \psi_n$$

matrix eigenvalue equation

[added
after 2017]

A projector operator $|a_n\rangle\langle a_n|$ is represented by a matrix

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} (0 \ 0 \ \dots \ 1 \ \dots \ 0) = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix}$$

$d \times 1$ $1 \times d$

observe $\sum |a_n\rangle\langle a_n| = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \mathbb{I}$
unit
mtx

Hermitian conjugate of an operator $\rightarrow$ transpose complex conj of matrix

$$|X\rangle = B|Y\rangle \Rightarrow \langle X| = \langle Y| B^\dagger$$

$$\langle a_n|X\rangle = \sum_m \langle a_n|B|a_m\rangle \langle a_m|Y\rangle \quad \langle X|a_n\rangle = \sum_m \langle Y|a_m\rangle \langle a_m|B^\dagger|a_n\rangle$$

$$\chi_n = \sum_m B_{nm} \psi_m$$

$$\chi_n^* = \sum_m \psi_m^* (B^\dagger)_{mn}$$

[matrix times col. vector]

$$\begin{aligned} \chi_n^* &= \sum_m B_{nm}^* \psi_m^* \\ &= \sum_m \psi_m^* (B^{*T})_{mn} \end{aligned}$$

$$\text{so } (B^\dagger)_{mn} = (B^{*T})_{mn}$$

Spin- $\frac{1}{2}$ system: matrix representation in the S_z basis

$$|+\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leftarrow \text{called a "spinor"}$$

$$B \rightarrow \begin{pmatrix} \langle +|B|+\rangle & \langle +|B|-\rangle \\ \langle -|B|+\rangle & \langle -|B|-\rangle \end{pmatrix}$$

$$S_z \rightarrow \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$

[diagonal as expected]
w/ eigenvalues

$$S_x \rightarrow \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$S_y \rightarrow \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}$$

$$S^2 \rightarrow \begin{pmatrix} \frac{3\hbar^2}{4} & 0 \\ 0 & \frac{3\hbar^2}{4} \end{pmatrix}$$

[exercise: show $[S_x, S_y] = i\hbar S_z$
also $S^2 = S_x^2 + S_y^2 + S_z^2$
as mtr eqns]

Define $S_i = \frac{\hbar}{2} \sigma_i$ $\sigma_i =$ Pauli spin matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Observe $\sigma_i^\dagger = (\sigma_i^T)^* = \sigma_i$ Hermitian matrices
(because S_i is an observable)

~~Exercise~~ exercise

$$a) [S_x, S_y] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\hbar S_z$$

$$[S_y, S_z] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\hbar S_x$$

$$[S_z, S_x] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\hbar S_y$$

$$b) \frac{\hbar^2}{4\sigma} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \frac{3\hbar^2}{4} \mathbb{1} \checkmark$$

Eigenvectors of S_z

We already know that

$$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$$

operator

matrix

$$S_z \rightarrow \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ket

col. vector

$$|+\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{check: } \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark$$

$|\pm\rangle$ are also eigenstates of S^2 w/ eigenvalue $\frac{3\hbar^2}{4}$

$$S^2 \rightarrow \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Complete set of simultaneous eigenstates
 $\Rightarrow S^2$ and S_z compatible

$$[S^2, S_z] = 0 \quad \text{because identity commutes w/ anything}$$

Eigenvectors of S_x

8-7

$$S_x |\psi\rangle = \lambda |\psi\rangle$$

Solve using matrix rep. in S_z -basis:

$$S_x \rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_x$$

$$|\psi\rangle \rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left(\frac{\hbar}{2} \sigma_x - \lambda \mathbb{1} \right) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Trivial solution ($\alpha = \beta = 0$) unless $\det = 0$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

no surprise

$$S_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

For $\lambda = +\frac{\hbar}{2}$ we have $\alpha = \beta \Rightarrow |\psi\rangle = \begin{pmatrix} \beta \\ \beta \end{pmatrix}$

Normalize $\langle\psi|\psi\rangle = (\beta^* \beta) \begin{pmatrix} \beta \\ \beta \end{pmatrix} = 2|\beta|^2 = 1 \Rightarrow \beta = \frac{1}{\sqrt{2}} e^{i\theta}$

overall phase arbitrary: choose $\theta = 0$

$$|S_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow \text{call this } |+\rangle$$

$$\text{Note } |+\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$$

For $\lambda = -\frac{\hbar}{2}$, we have $\alpha = -\beta \Rightarrow |\psi\rangle = \begin{pmatrix} -\beta \\ \beta \end{pmatrix}$

Again choose $\theta = 0$

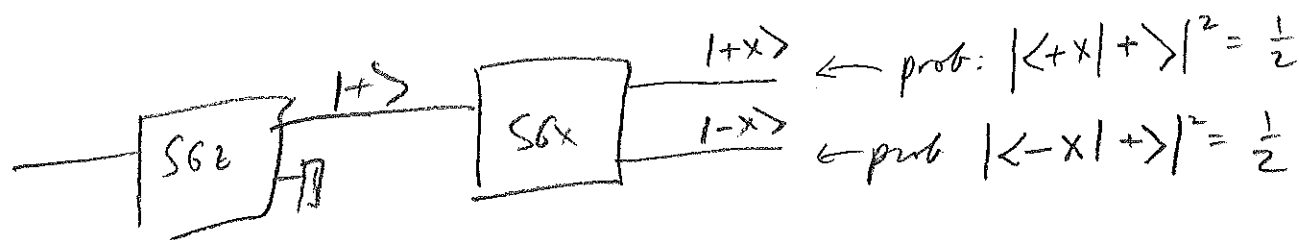
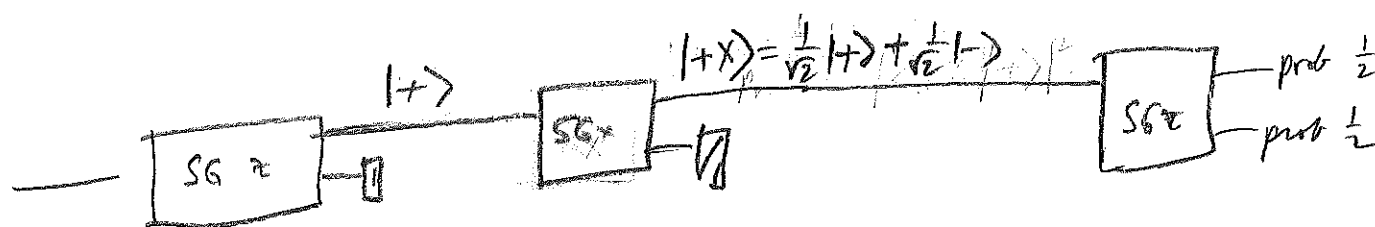
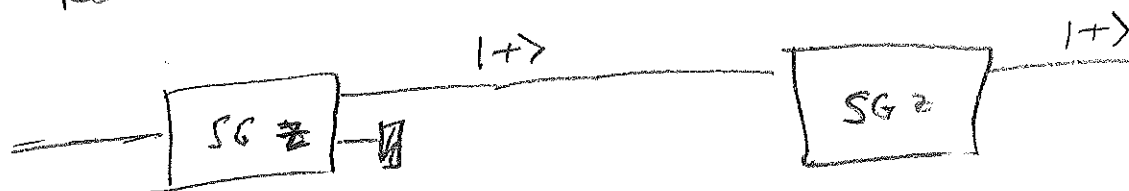
$$|S_x = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \leftarrow \text{call this } |-\rangle$$

$$|-\rangle = -\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$$

Note: I have a reason for choosing upper component negative: see ahead

- Because $[S_x, S_z] \neq 0$, $S_x + S_z$ are incompatible and so eigenstates $|\pm x\rangle$ are distinct from $|\pm\rangle \equiv |\pm z\rangle$
- Because $[S^2, S_x] = 0$ (since S^2 prop to identity) $S^2 + S_x$ are compatible and so $|\pm x\rangle$ are also S^2 eigenstates

Return to Stern Gerlach



$$\langle +x | + \rangle = \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle +x | = \frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - |$$