Review

Observable A — hermitian Operator $\hat{A}$

spectrum: possible results of measurements of A: $\{a_n\}$ eigenvalues of $\hat{A}$

real #'s → due to hermiticity

discrete or continuous

determinate states $|a_n\rangle$ eigenstates of $\hat{A}$

orthonormality $\langle a_n | a_m \rangle = \delta_{nm}$ → due to hermiticity

completeness $\sum |a_n\rangle \langle a_n| = \mathbb{1}$

indeterminate state $|\psi\rangle$ ~~not (...)~~ non-eigenstate

expansion in A-basis $|\psi\rangle = \sum |a_n\rangle \langle a_n|\psi\rangle = \sum \psi_n |a_n\rangle$

probability amplitude $\psi_n = \langle a_n|\psi\rangle$

probabilities $|\psi_n|^2 = \langle\psi|a_n\rangle\langle a_n|\psi\rangle$

expectation value $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$ if $\langle\psi|\psi\rangle = 1$

uncertainty $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

~~...~~ measurement → projection of state

$$|\psi\rangle \longrightarrow \hat{\Pi}_n|\psi\rangle = \psi_n |a_n\rangle$$

Compatibility

Two observables A and B are compatible if there exists a complete set of states $|\chi_n\rangle$ that have well-defined values for both A + B. (determinate state)

These states are therefore simultaneous eigenstates of $\hat{A}$ and $\hat{B}$

$$\hat{A}|\chi_n\rangle = a_n|\chi_n\rangle$$
$$\hat{B}|\chi_n\rangle = b_n|\chi_n\rangle$$

Claim: $\hat{A}$ and $\hat{B}$ commute

Proof: Consider an arbitrary state $|\psi\rangle \overset{\text{completeness}}{=} \sum_n c_n|\chi_n\rangle$

Consider
$$\hat{A}\hat{B}|\psi\rangle = \sum_n c_n \hat{A}\hat{B}|\chi_n\rangle$$
$$= \sum c_n \hat{A}\, b_n|\chi_n\rangle$$
$$= \sum c_n b_n \hat{A}|\chi_n\rangle$$
$$= \sum c_n b_n a_n|\chi_n\rangle$$

Similarly $\hat{B}\hat{A}|\psi\rangle = \sum c_n a_n b_n|\chi_n\rangle$

Thus $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$

Since this holds true for an arbitrary state $|\psi\rangle$ we conclude $\hat{A}\hat{B} = \hat{B}\hat{A}$.

Commutators

The commutator of $\hat{A}$ and $\hat{B}$ is $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

Compatible observables A + B $\Rightarrow$ $[\hat{A},\hat{B}] = 0$

[HW on commutators]

[Contrapositive] If $[\hat{A},\hat{B}] \neq 0$ then A + B are incompatible

If A, B are compatible there exist <u>some</u> states (i.e. the simultaneous eigenstates) for which both $\Delta A = 0$ and $\Delta B = 0$

$\therefore$ $\Delta A \, \Delta B = 0$ for these states

But Heisenberg uncertainty $\Delta x \, \Delta p \geq \frac{\hbar}{2}$ for all states

i.e. X and P are incompatible, and so we can't say $[X, P] = 0$

(Later we'll learn $[\hat{X}, \hat{P}] = i\hbar \hat{1}$.)

In HW, you will prove

$\boxed{\text{Generalized uncertainty relations}}$

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right|$$ [HW]

HW: $[A, B] = -[B, A]$

$[\alpha A + \beta B, C] = \alpha [A, C] + \beta [B, C]$

$[A, BC] = [A, B] C + B [A, C]$

Jacobi

$[A, B]$ is antihermitian if A, B Hermitian

From now on, drop hats from operators.
[distinction between observable + operator from context]

Recall from Stern Gerlach experiment that
S_z and S_x are incompatible.
(also S_z + S_y and S_y + S_x)

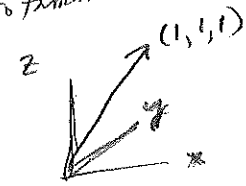
Therefore $[S_x, S_y] \neq 0$

Later, in HW, you'll prove that any form of angular momentum operator satisfies

$$\left.\begin{aligned} [J_x, J_y] &= i\hbar J_z \\ [J_y, J_z] &= i\hbar J_x \\ [J_z, J_x] &= i\hbar J_y \end{aligned}\right\} \text{angular momentum commutation relations}$$

<u>Comments</u>

- These follow from fact that angular momentum operators are generators of rotations, and rotations don't commute



- Eqns related by cyclic rotations

- $\hbar$ makes sense because $\hbar \to 0$ recovers classical physics

- i makes sense because S_i are hermitian but commutator of hermitian matrices is anti-hermitian (you'll prove this in HW)

[illegible]

[illegible]

Define "ladder operators"

$J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

Observe $J_\pm$ are not hermitian! $\quad J_\pm^\dagger = J_\mp$

Show $[J_z, J_\pm] = \pm\hbar J_\pm$ [HW]

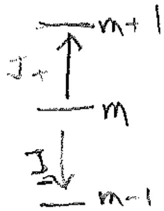
$$\Rightarrow \quad J_z J_\pm = J_\pm (J_z \pm \hbar)$$

$$\Rightarrow \quad J_z J_\pm^n = J_\pm^n (J_z \pm n\hbar) \qquad \text{[HW]}$$

(n any positive integer)

Let $|m\rangle$ be a (normalized) eigenstate of J_z with eigenvalue $m\hbar$

$$J_z J_\pm |m\rangle = J_\pm (m\hbar \pm \hbar)|m\rangle$$
$$= (m \pm 1)\hbar \, J_\pm |m\rangle$$



ie $J_\pm |m\rangle$ is an eigenstate of J_z w/ eigenvalue $(m\pm 1)\hbar$

J_+ is a raising operator: $\quad J_+|m\rangle \sim |m+1\rangle$

J_- is a lowering operator $\quad J_-|m\rangle \sim |m-1\rangle$

$J_\pm|m\rangle$ are not necessarily normalized, so $\quad \boxed{\begin{aligned} J_+|m\rangle &= c_+(m)\,|m+1\rangle \\ J_-|m\rangle &= c_-(m)\,|m-1\rangle \end{aligned}}$

Classically

$$\vec{J}^2 = \vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2$$

QM, direction doesn't make sense because cannot simultaneously define the diff. components of $\vec{J}$

Take our <u>definition</u> of the operator $\hat{J}^2$ to be $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

[HW] Show

$$J_- J_+ = J^2 - J_z^2 - \hbar J_z$$

$$J_+ J_- = J^2 - J_z^2 + \hbar J_z$$

Consider a spin-$\frac{1}{2}$ particle and let $J \to S$

There are only two eigenstates $|m\rangle$ of S_z:

$$m \uparrow \quad \begin{array}{l} \text{——} \quad |\tfrac{1}{2}\rangle = |+\rangle \\ \text{——} \quad |-\tfrac{1}{2}\rangle = |-\rangle \end{array} \qquad S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

Apply product to spin-$\frac{1}{2}$ eigenstates

$$S_- \underbrace{S_+|+\rangle}_{0} = S^2|+\rangle - \underbrace{S_z^2|+\rangle}_{\frac{\hbar^2}{4}|+\rangle} - \underbrace{\hbar S_z|+\rangle}_{\frac{\hbar^2}{2}|+\rangle}$$

because $|\frac{3}{2}\rangle$ doesn't exist

$$S^2|+\rangle = \frac{3}{4}\hbar^2|+\rangle$$

Similarly $$S_+\underbrace{S_-|-\rangle}_{0} = S^2|-\rangle - \underbrace{S_z^2|-\rangle}_{\frac{\hbar^2}{4}|-\rangle} + \underbrace{\hbar S_z|-\rangle}_{-\frac{\hbar^2}{2}|-\rangle}$$

because $|-\frac{3}{2}\rangle$ doesn't exist

$$S^2|-\rangle = \frac{3}{4}\hbar^2|-\rangle$$

$|\pm\rangle$ are both eigenstates of S^2
w/ eigenvalue $\frac{3}{4}\hbar^2$
(degenerate w.r.t. S^2 but not w.r.t. S_z)

S_z and S^2 are compatible (there exists a complete set of states ...)

$[\hat{S}_z, \hat{S}^2] = 0$

Prove $[J_z, J^2] = 0$ directly using comm relations [HW]

$$S_-|+\rangle = c_-(\tfrac{1}{2})\,|-\rangle$$

How do we determine $c_-(\tfrac{1}{2})$?

$$\langle +|\underbrace{S_-^\dagger}_{S_+} = \langle -|\, c_-^*(\tfrac{1}{2})$$

Now combine

$$\langle +|S_+ S_-|+\rangle = \langle -|\, c_-^*(\tfrac{1}{2})\, c_-(\tfrac{1}{2})\,|-\rangle = |c_-(\tfrac{1}{2})|^2$$

$$\hookrightarrow = \langle +|\, S^2 - S_z^2 + \hbar S_z\,|+\rangle$$

$$= \langle +|\; \frac{3\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 + \hbar\left(\frac{\hbar}{2}\right)|+\rangle$$

$$= \hbar^2 \underbrace{\langle +|+\rangle}_{1}$$

$$|c_-(\tfrac{1}{2})| = \hbar$$

$$\Rightarrow c_-(\tfrac{1}{2}) = \hbar e^{i\theta}$$

$$S_+|-\rangle = c_+(-\tfrac{1}{2})|+\rangle$$

$$\Rightarrow \quad c_+(-\tfrac{1}{2}) = \langle +| S_+ |-\rangle$$

$$= \langle -| S_+^\dagger |-\rangle^*$$

$$= \langle -| S_- |+\rangle^*$$

$$= \langle -| c_-(\tfrac{1}{2}) |-\rangle^*$$

$$= c_-^*(\tfrac{1}{2})$$

$$= \hbar e^{-i\theta}$$

Thus $\quad S_-|+\rangle = \hbar e^{i\theta}|-\rangle$

$\qquad\quad S_+|-\rangle = \hbar e^{-i\theta}|+\rangle$

Redefine $|-\rangle' = e^{i\theta}|-\rangle \quad \Rightarrow \quad S_-|+\rangle = \hbar|-\rangle'$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad S_+|-\rangle' = \hbar|+\rangle$

(still normalized + orthogonal to $|+\rangle$)

Now drop primes:

$$\boxed{\begin{aligned} S_-|+\rangle &= \hbar|-\rangle \\ S_+|-\rangle &= \hbar|+\rangle \end{aligned}}$$

[Alt, given $|+\rangle$, define $|-\rangle = \frac{1}{\hbar} S_-|+\rangle$

It is (1) an eigenstate
(2) normalized]

Claim: $S_- = \hbar \; |-\rangle\langle+|$

$S_+ = \hbar \; |+\rangle\langle-|$ Observe $S_+^\dagger = S_-$

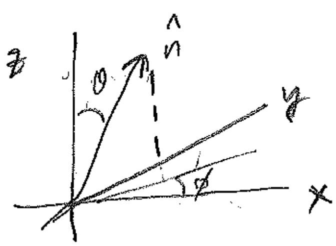
Prove by acting on an arbitrary ket $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ of both sides

Then $S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+| \right)$

$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+| \right)$

Recall also $\begin{cases} \hat{S}_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-| \right) \\ \hat{\mathbb{1}} = |+\rangle\langle+| + |-\rangle\langle-| \end{cases}$

Let $\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



Define $S_n \equiv \hat{n}\cdot\vec{S}$ (component of spin in $\hat{n}$ direction)

$= \sin\theta\cos\phi \, S_x + \sin\theta\sin\phi \, S_y + \cos\theta \, S_z$

We'll now learn how to represent operators as matrices.