

Hermitian conjugation

Given operator $\hat{A}$, there exists another operator $\hat{A}^\dagger$ (called hermitian conjugate of $\hat{A}$), such that

$$\text{if } \hat{A}|\psi\rangle = |\chi\rangle \quad \text{then} \quad \langle\chi| = \langle\psi| \hat{A}^\dagger$$

$$\text{Corollary: } \langle\psi| \hat{A}^\dagger |\phi\rangle = \langle\chi|\phi\rangle = \langle\phi|\chi\rangle^* = \langle\phi| \hat{A} |\psi\rangle^* \quad (*)$$

Claim: $(\hat{A}^\dagger)^\dagger = \hat{A}$ (proof)

If $\hat{A} = c\hat{I}$ then $\hat{A}^\dagger = c^* \hat{I}$ (proof) (hermitian conj analog of complex conj)

$$(\hat{B}\hat{A})^\dagger = \hat{A}^\dagger \hat{B}^\dagger \quad (\text{no unitary}) \quad (**)$$

Proof: Let $|\omega\rangle = \hat{B}\hat{A}|\psi\rangle$ then $\langle\omega| = \langle\psi| (\hat{B}\hat{A})^\dagger$

Now $|\omega\rangle = \hat{B}(\hat{A}|\psi\rangle) = \hat{B}|\chi\rangle \Rightarrow \langle\omega| = \langle\chi| \hat{B}^\dagger$

But $|\chi\rangle = \hat{A}|\psi\rangle \Rightarrow \langle\chi| = \langle\psi| \hat{A}^\dagger$ or

$$\langle\omega| = (\langle\psi| \hat{A}^\dagger) \hat{B}^\dagger = \langle\psi| \hat{A}^\dagger \hat{B}^\dagger \quad \text{QED}$$

$$\langle\phi| \hat{A} \hat{B} |\psi\rangle^* = \langle\psi| \hat{B}^\dagger \hat{A}^\dagger |\phi\rangle$$

← use (*) and (**)

hermitian conjugation $\Rightarrow$ ket $\leftrightarrow$ bra
 $A \rightarrow A^\dagger$

reverse order of objects

Hermitian operators

An operator $\hat{A}$ is Hermitian if $\hat{A}^\dagger = \hat{A}$
 (anti-Hermitian if $\hat{A}^\dagger = -\hat{A}$)

All observables correspond to Hermitian operators.
 (eg. energy $E \rightarrow$ Hamiltonian $\hat{H}$)

Two key results: a Hermitian operator has

① real eigenvalues

[measurements give real results]

② orthogonal eigenstates

We'll prove these using $\hat{H}$ but results are general.

• assume (for now) $\hat{H}$ has discrete eigenstates $|E_n\rangle$

[labeled by integers]

$$\hat{H}|E_n\rangle = E_n|E_n\rangle$$

Let $|E_n\rangle$ be normalized: $\langle E_n|E_n\rangle = 1$

① Consider $E_n = \langle E_n|\hat{H}|E_n\rangle$

$$E_n^* = \langle E_n|\hat{H}|E_n\rangle^*$$

$$= \langle E_n|\hat{H}^\dagger|E_n\rangle$$

$$= \langle E_n|\hat{H}|E_n\rangle$$

$$= E_n \quad \Rightarrow \quad E_n \in \mathbb{R}$$

② Eigenstates corresponding to distinct eigenvalues are orthogonal 6-3

Consider $\langle E_m | \hat{H} | E_n \rangle = E_n \langle E_m | E_n \rangle$

Take cc. $\langle E_n | \hat{H}^\dagger | E_m \rangle = E_n^* \langle E_n | E_m \rangle$

$$\langle E_n | \hat{H} | E_m \rangle = E_n \langle E_n | E_m \rangle$$

$$E_m \langle E_n | E_m \rangle = E_n \langle E_n | E_m \rangle$$

$$(E_m - E_n) \langle E_n | E_m \rangle = 0$$

Since $E_m \neq E_n \Rightarrow \langle E_n | E_m \rangle = 0$

If eigenstates are degenerate, proof fails.

But an orthogonal basis can always be chosen

by the Gram-Schmidt orthonormalization procedure

e.g. suppose $\langle E_n^1 | E_n^2 \rangle \neq 0$ when $\begin{cases} \hat{H} | E_n^1 \rangle = E_n | E_n^1 \rangle \\ \hat{H} | E_n^2 \rangle = E_n | E_n^2 \rangle \end{cases}$

Define $|\tilde{E}_n^2\rangle = |E_n^2\rangle - \langle E_n^1 | E_n^2 \rangle |E_n^1\rangle$

$\hat{H} |\tilde{E}_n^2\rangle = E_n |\tilde{E}_n^2\rangle$ still an eigenstate

$\langle E_n^1 | \tilde{E}_n^2 \rangle = 0$ ✓

Normalize $|\tilde{E}_n^2\rangle \Rightarrow |\hat{E}_n^2\rangle$

Then $|E_n^1\rangle$ and $|\hat{E}_n^2\rangle$ form an orthonormal eigenbasis

We can safely assert that there exists a basis γ

$$\langle E_n | E_m \rangle = \delta_{nm}$$

$$\delta_{nm} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$$

[problem]

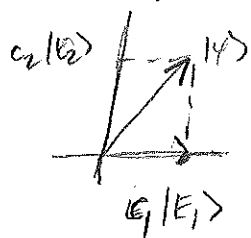
We assume that eigenstates of $\hat{H}$ are complete, i.e. any state $|\psi\rangle$ can be written as a linear combination

$$|\psi\rangle = \sum_n c_n |E_n\rangle$$

Since basis is orthonormal, $c_n = \langle E_n | \psi \rangle$.

$$\text{check: } \langle E_n | \psi \rangle = \sum_m c_m \langle E_n | E_m \rangle = \sum_m c_m \delta_{nm} = c_n$$

Define project operator $\hat{\Pi}_n$ as $\hat{\Pi}_n |\psi\rangle = c_n |E_n\rangle$
 Measurement of E_n collapses $|\psi\rangle$ to projection onto $|E_n\rangle$.



$$\text{Then } \hat{\Pi}_n = |E_n\rangle \langle E_n|$$

$$\text{Easy to verify: } \hat{\Pi}_n^2 = \hat{\Pi}_n$$

$$\hat{\Pi}_n \hat{\Pi}_m = 0 \quad n \neq m$$

$$\text{Completeness: } \sum_n \hat{\Pi}_n = \sum_n |E_n\rangle \langle E_n| = \mathbb{1}$$

← complete.
 if basis is not complete, can't build $|\psi\rangle$ back up out of $|E_n\rangle$

Provided $|\psi\rangle$ is normalized
 the Prob. of obtaining E_n is given by

$$\langle \psi | \hat{\Pi}_n | \psi \rangle = \underbrace{\langle \psi | E_n \rangle}_{c_n^*} \underbrace{\langle E_n | \psi \rangle}_{c_n} = |c_n|^2$$

check: the probabilities add up to 100%:

$$\sum_n |c_n|^2 = \sum_n \langle \psi | \hat{\Pi}_n | \psi \rangle = \langle \psi | \mathbb{1} | \psi \rangle = \langle \psi | \psi \rangle = 1$$

(HW) Show that $\hat{H} = \sum E_n \hat{\Pi}_n$

(Two operators are equal if they give the same result when acting on an arbitrary ket $|\psi\rangle$.)

(HW) Show $\langle\psi|\hat{H}|\psi\rangle = \langle E \rangle$

(HW) Show $\langle\psi|\hat{H}^2|\psi\rangle = \langle E^2 \rangle$

(HW) Show If $|\chi\rangle = (\hat{H} - \langle E \rangle)|\psi\rangle$, then $\langle\chi|\chi\rangle = (\Delta E)^2$

[Hint: we can insert $\hat{1} = \sum_n \hat{\Pi}_n$ at an arbitrary point
 or $\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{1}\hat{H}\hat{1}|\psi\rangle]$

An eigenstate of $\hat{H}$ has zero uncertainty. $\Delta E = 0$

Conversely, a state of zero uncertainty is an eigenstate

Proof: $\Delta E = 0 \Rightarrow \langle\chi|\chi\rangle = 0 \Rightarrow |\chi\rangle = 0$

$\Rightarrow \hat{H}|\psi\rangle = \langle E \rangle|\psi\rangle \Rightarrow \text{eigenstate}$