Linear operators

An operator $\hat{A}$ acts (from the left) on a ket to give another ket

$$\hat{A}|\psi\rangle = |\chi\rangle$$

A linear operator obeys:

$$\hat{A}\left[|\psi\rangle + |\phi\rangle\right] = \hat{A}|\psi\rangle + \hat{A}|\phi\rangle \qquad \text{(distributive law)}$$

$$\hat{A}\left[c|\psi\rangle\right] = c\,\hat{A}|\psi\rangle$$

Multiplication of operators

The product $\hat{B}\hat{A}$ acts associatively

$$(\hat{B}\hat{A})|\psi\rangle = \hat{B}(\hat{A}|\psi\rangle) = \hat{B}|\chi\rangle$$

[NB: operator on the right, closest to the ket, acts first]

In QM, products are generally not commutative

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}$$

commutator $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

Eigenvalue equation

$$\hat{A}|\psi\rangle = |\chi\rangle$$

If $|\chi\rangle$ is proportional to $|\psi\rangle$

i.e. if $\hat{A}|\psi\rangle = \lambda|\psi\rangle$

then $|\psi\rangle$ is an eigenstate (eigenket) of $\hat{A}$

and λ is the eigenvalue associated w/ $|\psi\rangle$

Notes:

$c|\psi\rangle$ is also an eigenstate (by linearity)
but is not independent of $|\psi\rangle$

if only one independent eigenstate is associated w/ λ,
then the eigenvalue is nondegenerate

# of independent eigenstates associated w/ λ
is the degree of degeneracy

Physics	meets	math

observable A	$\longleftrightarrow$	linear operator $\hat{A}$
spectrum $\{a\}$	$\longleftrightarrow$	eigenvalues $\{a\}$
determinate states $\|a\rangle$	$\longleftrightarrow$	eigenstates $\|a\rangle$

$$\hat{A}\,|a\rangle = a\,|a\rangle$$

Thus we <u>define</u> $\hat{S}_z$ by

$$\hat{S}_z\,|+\rangle = \frac{\hbar}{2}\,|+\rangle$$

$$\hat{S}_z\,|-\rangle = -\frac{\hbar}{2}\,|-\rangle$$

Linearity $\Rightarrow$ (we can determine action of $\hat{S}_z$ on any state)

$$\begin{aligned}
\hat{S}_z|\psi\rangle &= \hat{S}_z\left(\alpha|+\rangle + \beta|-\rangle\right) \\
&= \alpha\,\hat{S}_z|+\rangle + \beta\,\hat{S}_z|-\rangle \\
&= \alpha\,\frac{\hbar}{2}\,|+\rangle + \beta\left(-\frac{\hbar}{2}\right)|-\rangle \\
&= \frac{\hbar}{2}\left(\alpha|+\rangle - \beta|-\rangle\right)
\end{aligned}$$

When is $|\psi\rangle$ an eigenstate of $\hat{S}_z$? Iff $\alpha = 0$, or $\beta = 0$

indeterminate states are <u>not</u> eigenstates

DUAL SPACE

[Cohen-Tannoudji: not necessarily isomorphic to state space if ∞ dim'l]

Associated with each ket $|\phi\rangle$ in the state space is a "bra" $\langle\phi|$ in the dual vector space [space of bra's]

Operators are placed to the right: $\langle\phi|\hat{A}$

INNER PRODUCT

Combine bra $\langle\phi|$ with some ket $|\psi\rangle$ to get the inner product $\langle\phi|\psi\rangle$ ("bracket") which is a complex number

Properties of inner product

① linear: inner product of $\langle\phi|$ and $c|\psi\rangle$ is $c\langle\phi|\psi\rangle$

" " " $\langle\phi|$ and $|\psi_1\rangle+|\psi_2\rangle$ is $\langle\phi|\psi_1\rangle+\langle\phi|\psi_2\rangle$

② symmetric: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

implies $\langle\psi|\psi\rangle$ is real.

③ nonnegative: $\langle\psi|\psi\rangle > 0$ unless $|\psi\rangle = 0$

The bra associated w/ $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

is $\langle\psi| = \alpha^*\langle +| + \beta^*\langle -|$

Proof: for any $\langle\phi|$:

$\langle\phi|\psi\rangle = \alpha\langle\phi|+\rangle + \beta\langle\phi|-\rangle$ linearity of i.p.

$\langle\phi|\psi\rangle^* = \alpha^*\langle\phi|+\rangle^* + \beta^*\langle\phi|-\rangle^*$ c.c.

$\langle\psi|\phi\rangle = \alpha^*\langle +|\phi\rangle + \beta^*\langle -|\phi\rangle$ symmetry of i.p.

$\langle\psi| = \alpha^*\langle +| + \beta^*\langle -|$ remove $|\phi\rangle$ because $|\phi\rangle$ arbitrary

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Normalization

$\sqrt{\langle\psi|\psi\rangle}$ is the norm of $|\psi\rangle$

If $N = \langle\psi|\psi\rangle$, define $|\psi'\rangle = \frac{1}{\sqrt{N}}|\psi\rangle$ (N positive)

Then $\langle\psi'|\psi'\rangle = 1$ and $|\psi'\rangle$ is normalized.

We'll assume kets are normalized.

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orthogonality

If $\langle\phi|\psi\rangle = 0$ then we say $|\psi\rangle$ and $|\phi\rangle$ are orthogonal

[Hw: prove Schwarz inequality]

use $\langle x|x\rangle \geq 0$

where $|x\rangle = \langle\phi|\phi\rangle|\psi\rangle - \langle\phi|\psi\rangle|\phi\rangle$

Assume that $\hat{S}_z$ eigenstates $|+\rangle$ and $|-\rangle$ are normalized

$\langle +|+\rangle = 1, \quad \langle -|-\rangle = 1$ If not, normalize.

They are orthogonal Proof later

$\langle +|-\rangle = 0$

Therefore $|+\rangle$ and $|-\rangle$ form an <u>orthonormal basis</u> for the space of states.

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

Claim: the coefficients α, β are given by inner products

$\alpha = \langle +|\psi\rangle$ (only works if basis is orthonormal)

$\beta = \langle -|\psi\rangle$

Proof: $\langle +|\psi\rangle = \langle +|\left(\alpha|+\rangle + \beta|-\rangle\right)$

$= \alpha\langle +|+\rangle + \beta\langle +|-\rangle$ linearity of i.p.

$= \alpha \cdot 1 + \beta \cdot 0$ orthonormality of basis

$= \alpha$

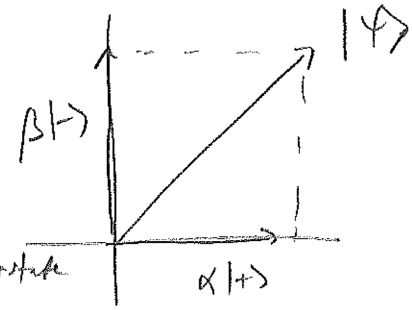
Projection operators

Given $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

Projection operators $\hat{\Pi}_\pm$ are defined by

$$\hat{\Pi}_+|\psi\rangle = \alpha|+\rangle$$

$$\hat{\Pi}_-|\psi\rangle = \beta|-\rangle$$



measurement projects the state onto eigenstate

Observe that $\hat{\Pi}_+^2 = \hat{\Pi}_+$ [repeated measurements]

$$\hat{\Pi}_-^2 = \hat{\Pi}_-$$

$$\hat{\Pi}_+\hat{\Pi}_- = 0$$

$$\hat{\Pi}_+ + \hat{\Pi}_- = \mathbb{1}$$

Claim: $\hat{\Pi}_+ = |+\rangle\langle+|$

$\hat{\Pi}_- = |-\rangle\langle-|$

bracket = complex # = inner product

ket bra = operator = outer product

$$\Pi_+|\psi\rangle = |+\rangle\langle+|\;|\psi\rangle = |+\rangle\;\langle+|\psi\rangle = |+\rangle\,\alpha \quad \text{etc}$$

associative axiom (Dirac)

check: $\hat{\Pi}_+^2 = |+\rangle\langle+|\;|+\rangle\langle+| = |+\rangle\underbrace{\langle+|+\rangle}_{1}\langle+| = |+\rangle\langle+| = \hat{\Pi}_+$

$\hat{\Pi}_+\hat{\Pi}_- = |+\rangle\underbrace{\langle+|-\rangle}_{0}\langle-| = 0$

[Sakurai, p.16]

$$\boxed{\text{Completeness relation: } |+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}}$$

Given $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

Prob. of obtaining $\pm\frac{\hbar}{2}$ can be expressed as $\langle\psi|\Pi_\pm|\psi\rangle$

check: $\langle\psi|\Pi_+|\psi\rangle = \langle\psi|+\rangle\langle+|\psi\rangle = |\langle+|\psi\rangle|^2 = |\alpha|^2$

prob

Also, the operator $\hat{S}_z$ can be expressed as

$$\hat{S}_z = \frac{\hbar}{2}\hat{\Pi}_+ + \left(-\frac{\hbar}{2}\right)\hat{\Pi}_-$$

$$= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

Proof: two operators are equal if they give the same result on an arbitrary ket.
Equivalently, if they give the same results on each of the elements of a (complete) basis
[since linear operators on an arbitrary ket can be expressed as linear comb. of actions on a basis]

$$\left(\frac{\hbar}{2}\hat{\Pi}_+ - \frac{\hbar}{2}\hat{\Pi}_-\right)|+\rangle = \frac{\hbar}{2}|+\rangle$$

$$\quad '' \quad |-\rangle = -\frac{\hbar}{2}|-\rangle \quad \text{same as } \hat{S}_z$$

prob.

problem

HW: Show $\langle S_z\rangle = \langle\psi|\hat{S}_z|\psi\rangle$

$\langle S_z^2\rangle = \langle\psi|\hat{S}_z^2|\psi\rangle$