Quantum mechanics

$\boxed{\text{State of a system}}$ (e.g. particle, atom, molecule, ping pong ball)

is described by a ket $|\psi\rangle \in$ complex vector space (Hilbert space)

bracket $\underbrace{\langle\phi|}_{\text{bra}}\underbrace{\psi\rangle}_{\text{ket}}$

vector space $\Rightarrow$ if $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states so are all complex linear combinations

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle \qquad c_i \in \mathbb{C}$$

[differs from classical physics]

Can write either $c|\psi\rangle$ or $|\psi\rangle c$

Postulates:

- all info about the state is contained in $|\psi\rangle$
- $|\psi\rangle$ and $c|\psi\rangle$ (for any $c \in \mathbb{C}$) describe the same physical state

observable A

Some measurable quantity of the system

eg S_z : z-component of spin

X : x-component of position

E : energy of a particle or of a system

Spectrum of the observable A

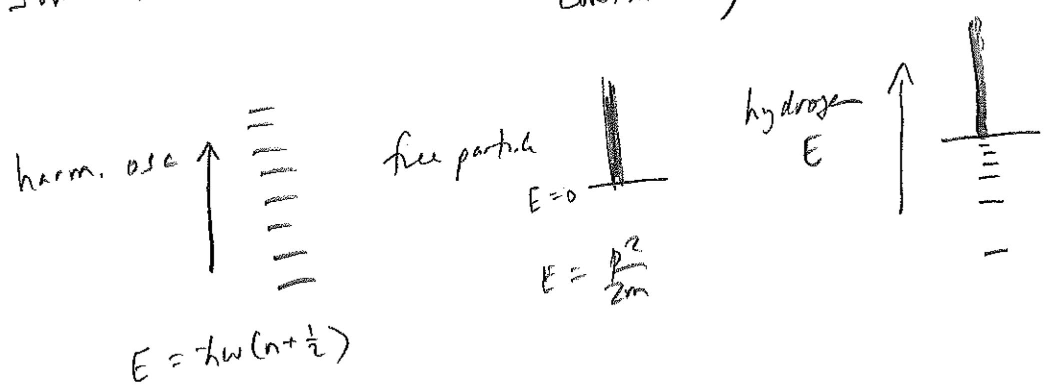
set of possible results of a measurement of A (always real numbers), $\{a\}$

[different meaning than spectrum emitted by an atom]

Some observables have discrete spectra (S_z: $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$)

Some observables have continuous spectra (X: $-\infty$ to ∞)

Some have both (E = discrete bound states, continuum)



$\boxed{\text{determinate states of an observable } A}$

Certain states $|a\rangle$, upon measurement of A, yield a definite (determinate) result

e.g. S_z has two determinate states ("eigenstates")

$$|S_z = \tfrac{\hbar}{2}\rangle = |+\rangle \quad \text{"spin up"}$$

$$|S_z = -\tfrac{\hbar}{2}\rangle = |-\rangle \quad \text{"spin down"}$$

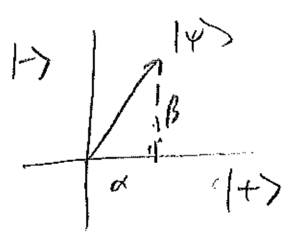
If measure S_z of $|+\rangle$ will always get $\frac{\hbar}{2}$

$|-\rangle$ " " " $-\frac{\hbar}{2}$

$\boxed{\text{indeterminate states}}$

Even though $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are the only possible <u>results</u> of a measurement of S_z, $|+\rangle$ and $|-\rangle$ are <u>not</u> the only possible <u>states</u>

Hilbert space contains also $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \qquad \alpha\beta \in \mathbb{C}$



$\leftarrow$ picture assumes $\alpha, \beta \in \mathbb{R}$, but $\alpha, \beta \in \mathbb{C}$ in general

If measure S_z of state $|\psi\rangle$, can get either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

(unless $\alpha = 0$, or $\beta = 0$)

$\boxed{\text{probability amplitudes}}$

α, β are called probability amplitudes because

probability of the result $\begin{cases} \frac{\hbar}{2} & \text{is } \dfrac{|\alpha|^2}{|\alpha|^2+|\beta|^2} \\ -\frac{\hbar}{2} & \text{is } \dfrac{|\beta|^2}{|\alpha|^2+|\beta|^2} \end{cases}$ where $|\alpha|^2 = \alpha^*\alpha$, $|\beta|^2 = \beta^*\beta$

If $\alpha = 1, \beta = 0$, $|\psi\rangle = |+\rangle \Rightarrow$ 100% prob of getting $\frac{\hbar}{2}$

Probabilities add to one:

$$\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2} + \frac{|\beta|^2}{|\alpha|^2+|\beta|^2} = 1$$

Since $|\psi\rangle$ and $c|\psi\rangle$ represent $\underline{\text{same}}$ physical state
we usually normalize

$$\alpha|+\rangle + \beta|-\rangle \longrightarrow \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left(\alpha|+\rangle + \beta|-\rangle\right) = \alpha'|+\rangle + \beta'|-\rangle$$

So that $|\alpha'|^2 + |\beta'|^2 = 1$ and prob of $\begin{cases} \frac{\hbar}{2} & \text{is } |\alpha'|^2 \\ -\frac{\hbar}{2} & \text{is } |\beta'|^2 \end{cases}$

Note that $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ give the same probabilities
(since same physical state)

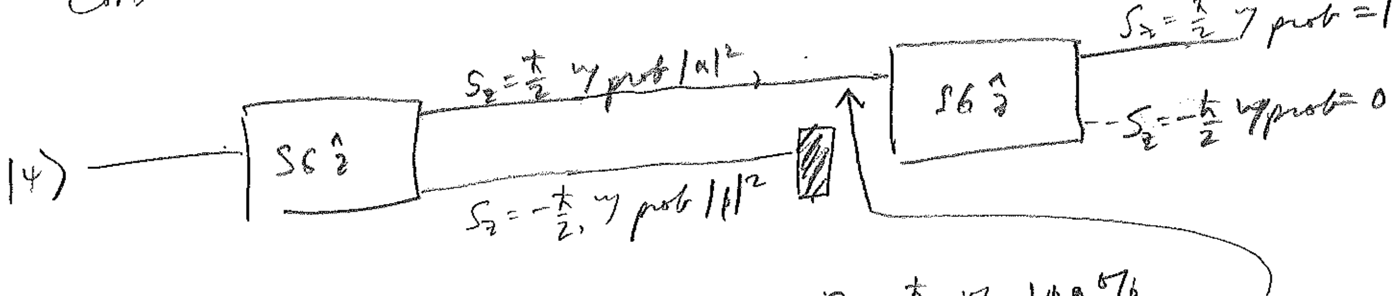
$$e^{i\theta}|\psi\rangle = e^{i\theta}\alpha|+\rangle + e^{i\theta}\beta|-\rangle$$

$\Downarrow$

prob $\frac{\hbar}{2}$ is $|e^{i\theta}\alpha|^2 = |\alpha|^2$

Measurement

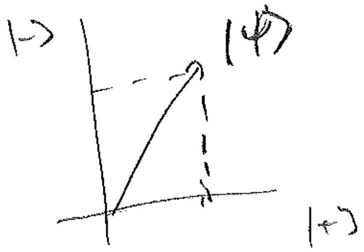
Consider an atom described by $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$



Because repeat measurement yields $S_z = \frac{\hbar}{2}$ w/ 100%, we infer that state going <u>into</u> 2nd apparatus was $|+\rangle$

∴ Act of measurement by 1st apparatus changes the state

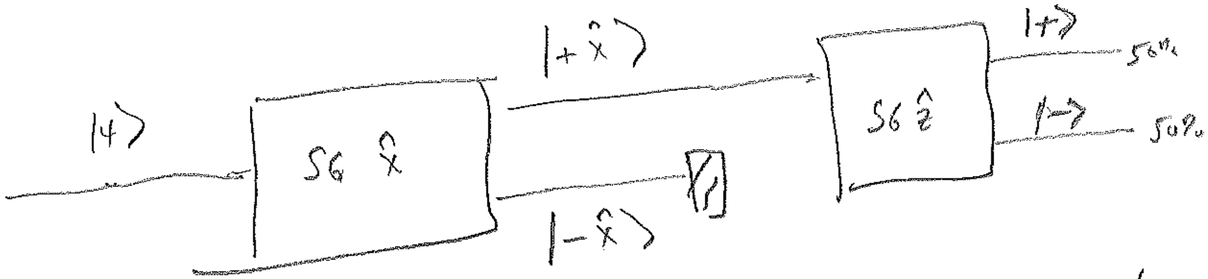
$$|\psi\rangle \xrightarrow[\text{if } S_z = \frac{\hbar}{2}]{} |+\rangle \quad \text{or} \quad \xrightarrow[\text{if } S_z = -\frac{\hbar}{2}]{} |-\rangle$$



Measurement <u>projects</u> $|\psi\rangle$ onto a determinate state

"collapse of the wavefunction"

<u>Copenhagen interpretation</u>



Let $|\pm\hat{x}\rangle$ be determinate states of S_x w/ results $\pm\frac{\hbar}{2}$

Second measurement suggests $|+\hat{x}\rangle = \alpha|+\rangle + \beta|-\rangle$

with $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$.

(Later we'll discover that $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

$|-\hat{x}\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$)

$\therefore$ determinate states of S_x are not the same as determinate states of S_z

$\Rightarrow$ S_x + S_z are incompatible

Measurement is not a deterministic process;
the same state can give different results.

[Feynman quote]

QM can only predict probabilities

[Einstein: "God does not play dice!"]

EPR (1936): QM is not complete
hidden variable theories

Bell's thm (1964): hidden variable theories
imply inequalities that are violated by QM
+ by expt

[Bohr: "stop telling God what to do"]

A philosopher once said, "It is necessary for the very existence of science that the same conditions always produce the same results." Well, they don't!

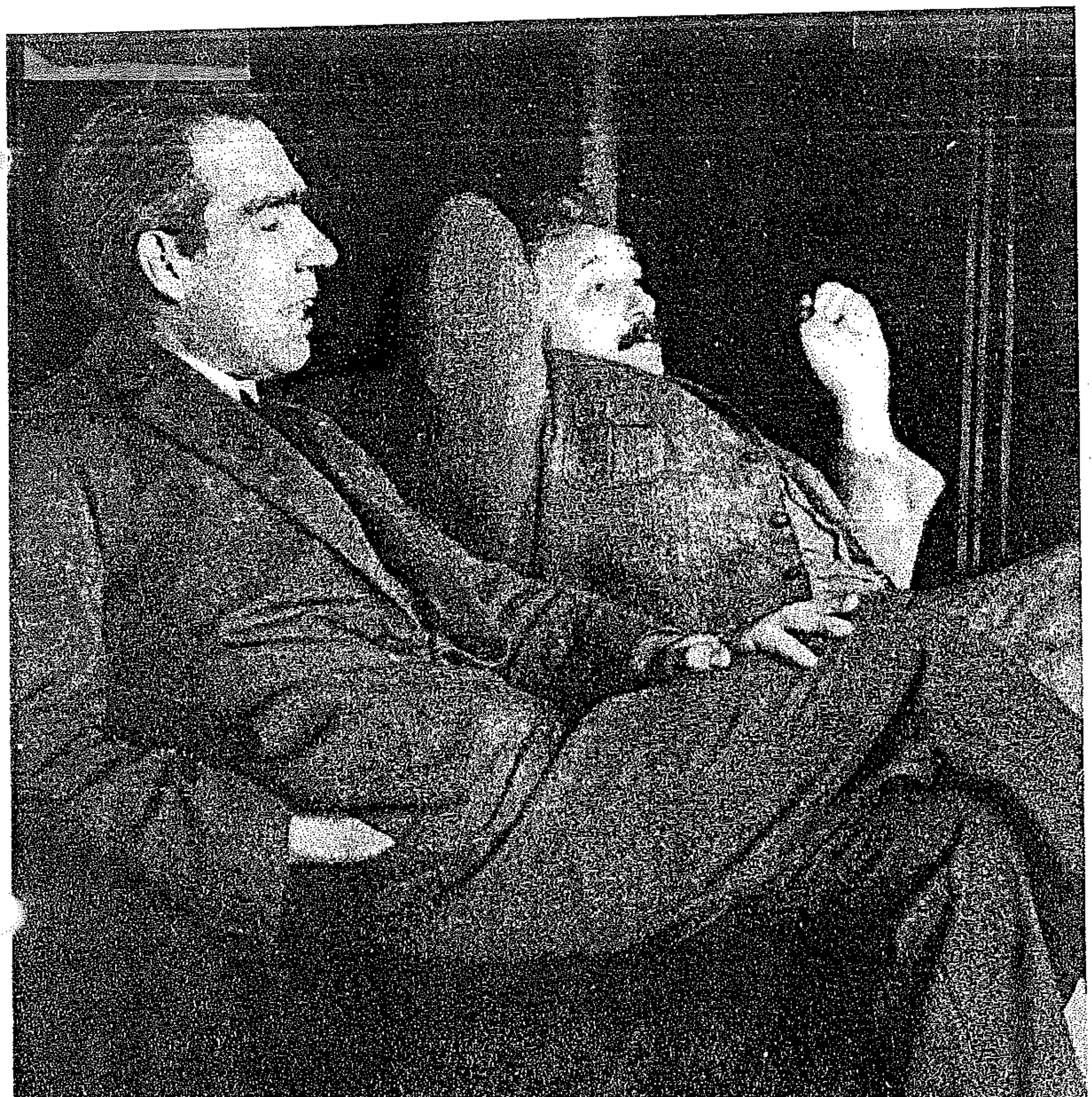
RICHARD FEYNMAN, *The Character of Physical Law* (1965)

"God does not play dice."

Albert Einstein

"Einstein, stop telling God what to do."

Niels Bohr



classical description

Spin state described by a classical vector $\vec{S}$ $\left[\longrightarrow \mathbb{R}^3 \text{ or } S^2 \text{ if } |\vec{S}| = \frac{\hbar}{2} \right]$

SG would give a continuous range of outcomes
but it does not

$$\mathbb{C}P^1 \cong S^2$$

QM description

Spin state described by a ket $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

SG gives probabilistic results
and projects the state

Subset of S^2 = all directions for which $f = 1$

power set of S^2
i.e. set of subsets
has cardinality
$\aleph_2$

Hidden variable alternative

Spin state described by $f(\theta, \phi)$

where $f = \pm 1$ for every angle

e.g. choosing 3 ~~angles~~ directions

we have 8 possibilities $(\pm\hat{a}, \pm\hat{b}, \pm\hat{c})$

exp rules this out

Completeness

Assume: the determinate states of an observable A constitute a complete basis for the space of all states

ie any state can be written as a linear combination of basis states

Suppose observable A has a finite no. of independent determinate states $|a_n\rangle$ $n=1,\dots d$

(eg $d=2$ for S_z, viz $|+\rangle$ and $|-\rangle$)

then an arbitrary state is

$$|\psi\rangle = \sum_{n=1}^{d} \psi_n |a_n\rangle$$

$d =$ dimension of state space

$\psi_n =$ coeff of $|\psi\rangle$ in A-basis

$=$ probability amplitude for measuring a_n

Probability that a measurement of $|\psi\rangle$ will yield a_n is $|\psi_n|^2$

Require $\sum_{n=1}^{d} |\psi_n|^2 = 1$. (Otherwise, normalize)

Expectation value

$\langle A \rangle$ = expectation value of A for the state $|\psi\rangle$

= mean value (weighted by probabilities) of results of measurement of A

$= \sum |\psi_n|^2 a_n$

Uncertainty

ΔA = uncertainty in A for the state $|\psi\rangle$

= root mean square deviation from $\langle A \rangle$

- results of measurements: a_n
- deviation from the mean of each measurement: $a_n - \langle A \rangle$

- weighted avg of square deviation $= \sum |\psi_n|^2 (a_n - \langle A \rangle)^2$

$$= \sum |\psi_n|^2 (a_n^2 - 2\langle A \rangle a_n - \langle A \rangle^2)$$

$$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle - \langle A \rangle^2$$

- mean square deviation $= \langle A^2 \rangle - \langle A \rangle^2$
- rms deviation $= \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

For a determinate state, $\Delta A = 0$ because measurement always gives one result a_n so deviation vanishes.

Converse is also true.

4-

HW:

show $\langle\psi|\psi\rangle = 1 \Rightarrow P_+ = |\alpha|^2$

calc $\langle S_z\rangle$, $\langle S_z^2\rangle$

show $\Delta S_z = \hbar|\alpha||\beta|$

explain that $\Delta S_z = 0 \Rightarrow |\psi\rangle = |+\rangle$ or $|-\rangle$