Reintroduce spin

electron spin $\vec{S}$ commutes w/ H, $\vec{L}$

Complete set of commuting ops $\{H, L^2, L_3, S^2, S_3\}$

mutual eigenstates $|n, l, m_l, s, m_s\rangle$

$$H\,|n, l, m_l, s, m_s\rangle = E_n\,|n, l, m_l, s, m_s\rangle, \quad E_n = -\frac{mc^2\alpha^2}{2n^2},\ n > 0$$

$$L^2\,|\ ''\ \rangle = \hbar^2 l(l+1)\,|\ ''\ \rangle, \quad 0 \le l \le n-1$$

$$L_3\,|\ ''\ \rangle = \hbar m_l\,|\ ''\ \rangle, \quad |m_l| \le l$$

$$S^2\,|\ ''\ \rangle = \tfrac{3}{4}\hbar^2\,|\ ''\ \rangle, \quad s = \tfrac{1}{2}$$

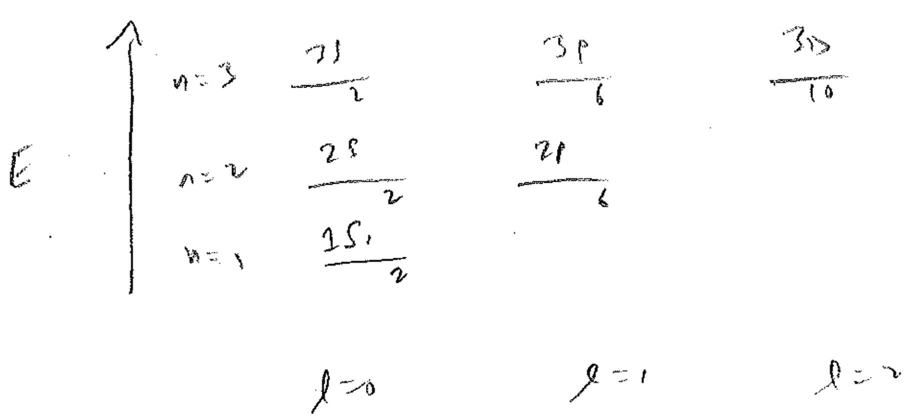
$$S_3\,|\ ''\ \rangle = \hbar m_s\,|\ ''\ \rangle, \quad m_s = \pm\tfrac{1}{2}$$

Define χ_{m_s} as $\begin{cases} \chi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix} \\ \chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix} \end{cases}$

$\rho = \frac{r}{na_0}$

In position space, eigenstates are

$$u_{nlm_lm_s}(r,\theta\,\phi) = (\text{const})\, r^l\, L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right) e^{-\frac{r}{na_0}}\, Y_{lm_l}(\theta,\phi)\,\chi_{m_s}$$



Recall: total angular momentum $\vec{J} = \vec{L} + \vec{S}$

J^2 commutes w/ L^2 + S^2 but not w/ L_z + S_z (unless $l=0$)

Choose complete set of commuting ops $\{H, L^2, S^2, J^2, J_z\}$

w/ associated eigenstates $|n, l, s, j, m_j\rangle$

in position space $\psi_{n,l,j,m_j}(r,\theta,\phi)$

How relate ψ_{nljm_j} to $u_{nlm_lm_s}$?

spatial $\otimes$ (spin $\frac{1}{2}$)

$|n, l, m_l, s, m_s\rangle \to |m_l; m_s\rangle$

$\underline{2P} \to \underline{2P_{3/2}}\ \underline{2P_{1/2}}$

Spin $(l+\frac{1}{2}) \oplus$ Spin $(l-\frac{1}{2})$

$|n, l, s, j, m_j\rangle \to |j, m_j\rangle$

ie $j = l+\frac{1}{2}$ and $l-\frac{1}{2}$

Use clebsch gordan to relate these [Schumacher p 309; Gasiorowicz]

$$\psi_{n,l,l+\frac{1}{2},m_j} = \begin{pmatrix} \sqrt{\dfrac{l+\frac{1}{2}+m_j}{2l+1}}\ u_{n,l,m_j-\frac{1}{2}} \\ \\ \sqrt{\dfrac{l+\frac{1}{2}-m_j}{2l+1}}\ u_{n,l,m_j+\frac{1}{2}} \end{pmatrix}$$

$$\psi_{n,l,l-\frac{1}{2},m_j} = \begin{pmatrix} -\sqrt{\dfrac{l+\frac{1}{2}-m_j}{2l+1}}\ u_{n,l,m_j-\frac{1}{2}} \\ \\ \sqrt{\dfrac{l+\frac{1}{2}+m_j}{2l+1}}\ u_{n,l,m_j+\frac{1}{2}} \end{pmatrix}$$

Addition of orbital and spin angular momentum

$$J^2 = L^2 + S^2 + 2L_z S_z + L_+ S_- + L_- S_+$$

$$L_\pm Y_{\ell m} = \sqrt{(\ell \pm m + 1)(\ell \mp m)}\,\hbar\, Y_{\ell, m\pm 1}$$

$$S_\pm \chi_\mp = \hbar \chi_\pm$$

$$\psi_{j, m+\frac{1}{2}} = \alpha Y_{\ell m} \chi_+ + \beta Y_{\ell, m+1} \chi_-$$

$$\begin{aligned} J^2 \psi_{j, m+\frac{1}{2}} &= \alpha \left[\ell(\ell+1) + \tfrac{3}{4} + 2m\left(\tfrac{1}{2}\right) \right] \hbar^2 Y_{\ell m} \chi_+ \\ &+ \alpha \left[\sqrt{(\ell+m+1)(\ell-m)} \right] \hbar^2 Y_{\ell, m+1} \chi_- \\ &+ \beta \left[\ell(\ell+1) + \tfrac{3}{4} + 2(m+1)\left(-\tfrac{1}{2}\right) \right] \hbar^2 Y_{\ell, m+1} \chi_- \\ &+ \beta \sqrt{(\ell-m)(\ell+m+1)}\, \hbar^2 Y_{\ell, m} \chi_+ \\ &= j(j+1)\hbar^2 \left[\alpha Y_{\ell m} \chi_+ + \beta Y_{\ell, m+1} \chi_- \right] \end{aligned}$$

$$\Rightarrow \alpha\left(\ell^2 + \ell + \tfrac{3}{4} + m\right) + \beta \sqrt{(\ell-m)(\ell+m+1)} = \alpha\, j(j+1)$$

$$\alpha \sqrt{(\ell+m+1)(\ell-m)} + \beta\left(\ell^2 + \ell - \tfrac{1}{4} - m\right) = \beta\, j(j+1)$$

$j = \ell + \frac{1}{2} \Rightarrow$ $\beta \sqrt{(\ell-m)(\ell+m+1)} = \alpha(\ell - m)$

$$\psi_{\ell+\frac{1}{2}, m+\frac{1}{2}} = \sqrt{\frac{\ell+m+1}{2\ell+1}}\, Y_{\ell m} \chi_+ + \sqrt{\frac{\ell-m}{2\ell+1}}\, Y_{\ell, m+1} \chi_-$$

$j = \ell - \frac{1}{2} \Rightarrow$ $\beta \sqrt{(\ell-m)(\ell+m+1)} = -(m+\ell+1)\alpha$

$$\psi_{\ell-\frac{1}{2}, m+\frac{1}{2}} = \sqrt{\frac{\ell-m}{2\ell+1}}\, Y_{\ell m} \chi_+ - \sqrt{\frac{m+\ell+1}{2\ell+1}}\, Y_{\ell, m+1} \chi_-$$

$n = 2$

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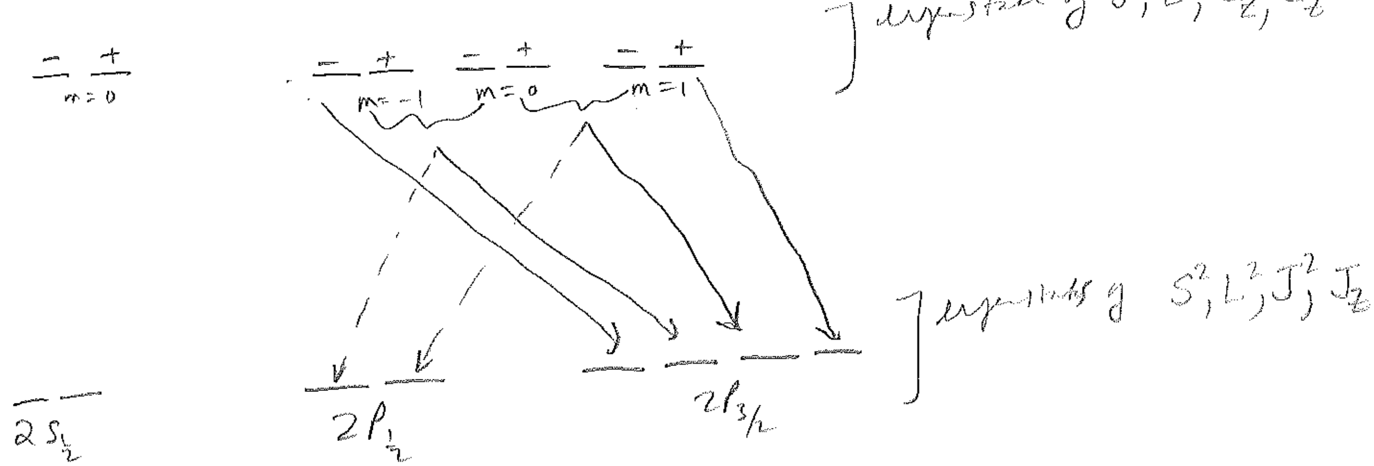

Recall from problem

$$\begin{array}{ll} j\ m_j & m_\ell\ m_s \end{array}$$

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1;\tfrac{1}{2}\rangle = Y_{11}\,\chi_{\frac{1}{2}}$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1;-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,Y_{10}\,\chi_{\frac{1}{2}} + \sqrt{\tfrac{1}{3}}\,Y_{11}\,\chi_{-\frac{1}{2}}$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|-1;\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |-1;-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|-1;-\tfrac{1}{2}\rangle$$