

Hydrogen atom Particle in central potential

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Reduction of 2-body problem to 1-body problem

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2$$

$$V = V(\vec{r}_1 - \vec{r}_2)$$

Let $\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$

$$M = m_1 + m_2$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$$

m = reduced mass

Then $T = \frac{1}{2} M \dot{\vec{r}}_{cm}^2 + \frac{1}{2} m \dot{\vec{r}}^2$

$$V = V(\vec{r})$$

$$H = \underbrace{\frac{\vec{p}_{cm}^2}{2M}}_{\text{"free particle" (hydrogen cm)}} + \underbrace{\frac{\vec{p}^2}{2m} + V(\vec{r})}_{\text{particle in 3d potential}}$$

If $m_1 \gg m_2$, then $m \approx m_2$ e.g. electron

skipped in
2013
+ after

$|E, l, m\rangle = \text{simultaneous eigenstate of } H, L^2, L_z$

$$u_{Elm}(r, \theta, \phi) = R_{Elm}(r) Y_{lm}(\theta, \phi)$$

$$H u_{Elm} = E u_{Elm}$$

$$H = -\frac{\hbar^2}{2m_e} \nabla^2 + V(r)$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \underbrace{\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)}$$

You will show this $= -\frac{L^2}{\hbar^2}$

$$-\frac{\hbar^2}{2m_e} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{L^2}{\hbar^2 r^2} \right] R_{Elm} Y_{lm} + V(r) R_{Elm} Y_{lm} = E R_{Elm} Y_{lm}$$

$$-\frac{\hbar^2}{2m_e} \left[\underbrace{\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R_{Elm}}{\partial r} \right)} - \frac{l(l+1)}{r^2} R_{Elm} \right] + V(r) R_{Elm} = E R_{Elm}$$

$$= \frac{1}{r} \frac{\partial^2}{\partial r^2} (r R_{Elm}), \text{ easy to verify}$$

$$\frac{d^2}{dr^2} (r R_{Elm}) - \frac{l(l+1)}{r^2} (r R_{Elm}) = -\frac{2m_e}{\hbar^2} (E - V(r)) (r R_{Elm})$$

radial Schrödinger eqn

N.B. Coefficients of the eqn depend on l , but not m .
 therefore eigenvalues E & eigenfunctions $R(r)$ do not
 depend on m i.e. the $(2l+1)$ eigenfunctions

$$u_{Elm}(r, \theta, \phi) = R(l) Y_{lm}(\theta, \phi) \quad m = -l, \dots, l$$

are degenerate $\Rightarrow$ this is due to spherical symmetry

[Drop m from label for $R_{El}(r)$]

$$u_{Elm}(r, \theta, \phi) = R_{El}(r) Y_{lm}(\theta, \phi)$$

E only depends on l