Eigenstates of H, L^2, L_3 in position space (spherical coords)

$$u(r,\theta,\phi) = \langle \vec{x} \,|\, E, l, m \rangle$$

Try separable ansatz $u(r,\theta,\phi) = R(r)\, Y_{lm}(\theta,\phi)$

Let $Y_{lm}(\theta,\phi) = g(\theta) f(\phi)$

$$\mathbb{L}_3 u = \hbar m u$$

$$\frac{\hbar}{i}\frac{df}{d\phi} = \hbar m f$$

$$\frac{df}{f} = i m \, d\phi$$

$$f = (\text{const})\, e^{im\phi}$$

we know that l, m = integer or half-integer
if m = integer then $f(\phi+2\pi) = f(\phi)$ because $e^{2\pi i m} = 1$
if m = half integer $f(\phi+2\pi) = -f(\phi)$

wavefunction is a function of position so $f(\phi+2\pi) = f(\phi)$

not single valued fn of point

$$e^{im(\phi+2\pi)} = e^{im\phi}$$

$$e^{2\pi i m} = 1$$

spin can be int or half int but orb. ang mom. must be integer

$m \in \mathbb{Z}$ [not half-integer]

$\therefore\ l = m_{max} \in \mathbb{Z}$

Bohr quantization postulate!

$$\Rightarrow Y_{lm}(\theta,\phi) = g(\theta)\, e^{im\phi}, \quad m = -l, \ldots, l$$

Rec.

$$L_\pm = \hbar e^{\pm i\phi}\left(\pm \frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)$$

$L_+ \uparrow$ — Y_{ll}

— $Y_{l,l-1}$

— $\vdots$

— $Y_{l,-l}$

$$L_+ Y_{ll} = 0$$

$$Y_{ll}(\theta,\phi) = g(\theta) e^{il\phi}$$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(g(\theta)e^{il\phi}\right) = 0$$

$$\frac{dg}{d\theta} + i\cot\theta\,(il)\,g(\theta) = 0$$

$$\frac{dg}{g} = l\cot\theta\, d\theta = l\,\frac{\cos\theta\, d\theta}{\sin\theta}$$

$$\ln g = l \ln\sin\theta + \text{const}$$

$$g(\theta) = C(\sin\theta)^l$$

$$\Rightarrow Y_{ll} = C\,(\sin\theta)^l e^{il\phi}$$

$Y_{lm}(\theta,\phi)$ are normalized by

$$\int |Y_{lm}(\theta,\phi)|^2 \overbrace{d\Omega}^{\text{solid angle}} = 1$$

$$\int |Y_{lm}(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi = 1$$

$$\int |Y_{ll}|^2 \sin\theta \, d\theta \, d\phi = |C|^2 \underbrace{\int_0^{\pi} \sin^{2l+1}\theta \, d\theta}_{\frac{2(2l)!!}{(2l+1)!!}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

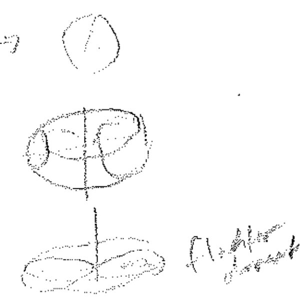
$$\Rightarrow C = \sqrt{\frac{(2l+1)!!}{(2l)!!} \cdot \frac{1}{4\pi}} \cdot \underbrace{(-1)^l}_{\text{convention}}$$

$$Y_{00} = +\frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$$

$\vdots$



$$2\frac{(2l)!!}{(2l+1)!!}$$

$$\int_0^{\pi} \sin^{2l+1}\theta \, d\theta = \int_{-1}^{1} (1-x^2)^l \, dx$$

$$\cancel{\binom{l}{m}} =$$

$$\| $$

$$\boxed{2 \sum_{m=0}^{l} (-1)^m \frac{l!}{m!(l-m)!} \frac{1}{(2m+1)}} = \begin{cases} l=0: & 2 \\ l=1: & \frac{4}{3} \\ l=2: & \frac{16}{15} \end{cases}$$

Recall $L_\pm Y_{\ell m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\ \hbar\, Y_{\ell, m\pm 1}$

$$Y_{\ell,\ell-1} = \frac{1}{\sqrt{2\ell}} \frac{L_-}{\hbar} Y_{\ell,\ell}$$

$$= \frac{e^{-i\phi}}{\sqrt{2\ell}} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right) C (\sin\theta)^\ell e^{i\ell\phi}$$

$$= \frac{e^{-i\phi} C}{\sqrt{2\ell}} \left(-\ell (\sin\theta)^{\ell-1} \cos\theta - \ell (\cos\theta)(\sin\theta)^{\ell-1} \right) e^{i\ell\phi}$$

$$Y_{\ell,\ell-1} = -\sqrt{2\ell}\, C\, (\sin\theta)^{\ell-1} (\cos\theta)\, e^{i(\ell-1)\phi}$$

etc

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$$\int |Y_{lm}|^2 \sin\theta \, d\theta \, d\phi = 1$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$ $l=0$ S state (sharp)

convention

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

$l=1$ P state (principle)

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$Y_{2,-1} = +\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{-i\phi}$$

$$Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{-2i\phi}$$

$l=2$ D state (diffuse

spdfgh

(fundamental)

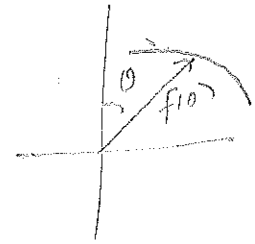
avoid this

Griffiths (2e) 139

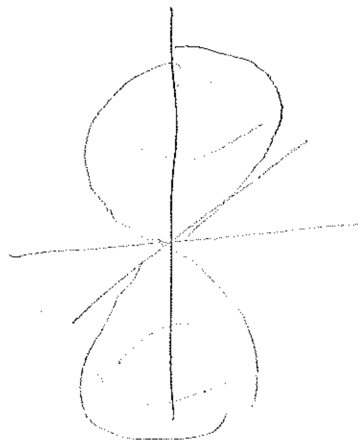
$$\boxed{Y_{l,-m} = (-1)^m Y_{l,m}^*}$$

$$\left[Y_{lm} = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi \, (l+|m|)!}} \, P_l^{|m|}(\cos\theta) \, e^{im\phi} \qquad \epsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases} \right]$$

visualize using polar ~~plot~~ plots (explain)



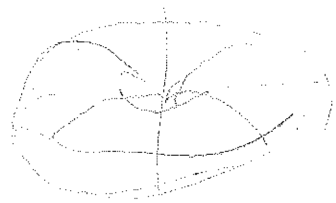
$|Y_{10}|^2 \sim \cos^2\theta$



← "p_z"

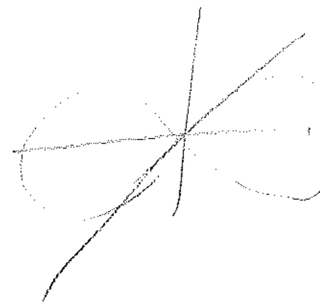
$|Y_{11}|^2 \sim \sin^2\theta$

$\sim |Y_{1,-1}|^2$



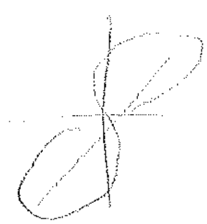
$Y_{1,1} - Y_{1,-1} \sim \sin\theta\,(e^{i\phi} + e^{-i\phi}) \sim 2\sin\theta\cos\phi$

$|Y_{11} - Y_{1,-1}|^2 \sim \sin^2\theta\cos^2\phi$



p_x

$|Y_{11} + Y_{1,-1}|^2 \sim \sin\theta\sin^2\phi$



p_y