Angular momentum review

Recall that rotations are generated by angular momentum op. $\vec{J}$

$$U = e^{-\frac{i\alpha \hat{n}\cdot\vec{J}}{\hbar}}$$ for a rotation through angle α about $\hat{n}$.

We showed this implies

$[J_1, J_2] = i\hbar J_3$ plus cyclic perms

For a particle w/ spin, $\vec{J} = \vec{L} + \vec{S}$

For a particle w/o spin, $\vec{J} = \vec{L}$ so

$[L_1, L_2] = i\hbar L_3$ plus cyclic perms

Classically $\vec{L} = \vec{x} \times \vec{p}$

$$L_3 = x_1 p_2 - x_2 p_1$$

$$\Downarrow$$

Quantum: $\hat{L}_3 = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1$

HW: Show using $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ that

$[L_1, L_2] = i\hbar L_3$ and cyclic perms

Orbital ang. mom. acts on position space wavefunctions.

In position space: $L_3 = \frac{\hbar}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$

Recall spherical coords

$x_3 = r\cos\theta$
$x_1 = r\sin\theta\cos\phi$
$x_2 = r\sin\theta\sin\phi$

I claim: $L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

First proof: chain rule $\frac{\partial}{\partial\phi} = \frac{\partial x_1}{\partial\phi}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial\phi}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial\phi}\frac{\partial}{\partial x_3}$

$$= \underbrace{-r\sin\theta\sin\phi}_{x_2}\frac{\partial}{\partial x_1} + \underbrace{r\sin\theta\cos\phi}_{x_1}\frac{\partial}{\partial x_2} + 0$$

$$= x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \quad \checkmark$$

Second proof: $U = e^{-\frac{i\alpha L_3}{\hbar}} = e^{-\alpha\frac{\partial}{\partial\phi}}$

$$U\psi(r,\theta,\phi) = e^{-\alpha\frac{\partial}{\partial\phi}}\psi(r,\theta,\phi)$$

$$= \left[1 - \alpha\frac{\partial\psi}{\partial\phi} + \frac{1}{2}(-\alpha)^2\frac{\partial^2\psi}{\partial\phi^2} + \dots\right]$$

$$= \psi(r,\theta,\phi-\alpha)$$

so U rotates wavefunction through angle α about x_3-axis

[HW]: show that

$$L_1 = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right)$$

$$L_2 = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right)$$

This implies

$$L_\pm = L_1 \pm i L_2 = i\hbar \left(\underbrace{[\sin\phi \mp i \cos\phi]}_{\mp i e^{\pm i\phi}} \frac{\partial}{\partial\theta} + \cot\theta \underbrace{[\cos\phi \pm i \sin\phi]}_{e^{\pm i\phi}} \frac{\partial}{\partial\phi} \right)$$

$$\boxed{L_\pm = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right)}$$

Save this for later

Consider a particle in a spherically-symmetric potential $V(r)$ (a.k.a. $\underline{\text{central}}$ potential because $\vec{F} = -\vec{\nabla} V$ points toward or away from the origin)

$$H = \frac{P^2}{2m} + V(x) \longrightarrow -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

In spherical coordinates

$$\nabla^2 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)}_{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2} \underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right)}_{\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Claim: spherical symmetry of potential $\Rightarrow [H, \vec{L}] = 0$

First consider $[H, L_3]$

$$\frac{\partial}{\partial\phi}\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\frac{\partial}{\partial\phi}$$

so $[H, L_3] = 0$

Not so easy to see in spherical coords that $[H, L_1] = 0$ and $[H, L_2] = 0$.
(unless write H in cartesian coords, use $[H, L_3]$ plus cyclic perms to show the others)

more general argument:

H is invariant under rotations because
$V(r)$ only depends on r and ∇^2 is a scalar operator

Let $H|E\rangle = E|E\rangle$

Then $U|E\rangle$ is also an eigenstate of same energy E
(because rotating the state will produce another state with same energy)

$$H\,(U|E\rangle) = E\,(U|E\rangle)$$

or

$$HU|E\rangle = UE|E\rangle = UH|E\rangle$$

$$(HU - UH)|E\rangle = 0$$

$$[H, U]|E\rangle = 0$$

Because energy eigenstates are a complete basis
$$[H, U]|\psi\rangle = 0 \quad \text{for any } |\psi\rangle$$
Because this holds for any $|\psi\rangle$

$$[H, U] = 0$$

$$U = e^{-\frac{i\alpha \hat{n}\cdot\vec{J}}{\hbar}} = 1 - \frac{i\alpha}{\hbar}\hat{n}\cdot\vec{J} + \dots$$

$$\Rightarrow [H, \hat{n}\cdot\vec{J}] = 0$$

$$[H, \vec{J}_i] = 0$$

Particle w/o spin
$$[H, \vec{L}_i] = 0 .$$

$[L_i, L_j] \neq 0$ so diff comps of orb. ang mom not compatible

However $[L^2, L_i] = 0$ where $L^2 = L_1^2 + L_2^2 + L_3^2$

[proved earlier in problem set]

Also $[H, L^2] = \sum_i [H, L_i^2] = \sum([H, L_i]L_i + L_i[H, L_i]) = 0$

$\Rightarrow$ H, L^2, and <u>one</u> component of $\vec{L}$ are mutually compatible

choose $L_3 \Rightarrow$ $\{H, L^2, L_3\}$ = set of mutually commuting op's.

Goal: find simultaneous eigenstates $|E, l, m\rangle$

$H|E, l, m\rangle = E\,|E, l, m\rangle$
$L^2|E, l, m\rangle = \hbar^2 l(l+1)\,|E, l, m\rangle$
$L_3|E, l, m\rangle = \hbar m\,|E, l, m\rangle$

Note: because $[H, L_\pm] = 0$, $|E, l, m\rangle$ will be degenerate for all m