[after they've done the problem]

Harmonic oscillator summary

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 \qquad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

operator approach (Dirac)

$$\begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}}\, X + \frac{i}{\sqrt{2m\hbar\omega}}\, P \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, X - \frac{i}{\sqrt{2m\hbar\omega}}\, P \end{cases} \qquad \Rightarrow [a, a^\dagger] = \mathbb{1}$$

$$\Rightarrow H = \hbar\omega\left(N + \tfrac{1}{2}\mathbb{1}\right) \quad \text{where} \quad N = a^\dagger a.$$

Let $|n\rangle$ be eigenstates of N w/ eigenvalue n + therefore eigenstate of H w/ eigenvalue $\hbar\omega(n + \frac{1}{2})$

You showed $a^\dagger$ + a are raising + lowering operators.

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \Rightarrow \quad a|0\rangle = 0$$

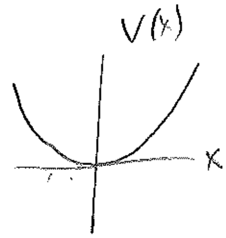
$$a\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad N|0\rangle = 0\,|0\rangle$$

$$H|0\rangle = \underbrace{\tfrac{1}{2}\hbar\omega}_{\text{grd st energy}}\,|0\rangle$$

mnemonic: $\sqrt{\text{larger integer}}$

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\,|0\rangle$$

Harmonic oscillator
(eg diatomic molecule)



$$E = \frac{p^2}{2m} + \frac{1}{2} k_s x^2$$

Natural frequency $\omega = \sqrt{\frac{k_s}{m}}$ $\Rightarrow$ $E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 = \hbar\omega \left[\frac{m\omega}{2\hbar} \hat{X}^2 + \frac{1}{2m\hbar\omega} \hat{p}^2 \right]$$

not necessary as done in practice

Operator approach

Define $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\, \hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\, \hat{p}$ non hermitian! (not an observable)

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, \hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\, \hat{p}$

$[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$ using $[\hat{x}, \hat{p}] = i\hbar\, \mathbb{1}$

Define $\hat{N} = \hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar} \hat{x}^2 + \frac{1}{2m\hbar\omega} \hat{p}^2 - \frac{1}{2} \hat{\mathbb{1}} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2} \hat{\mathbb{1}}$

$$\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \hat{\mathbb{1}}\right)$$

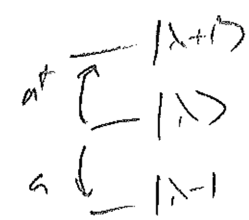
$[N, a^\dagger] = a^\dagger \Rightarrow N a^\dagger = a^\dagger (N+1)$

$[N, a] = -a \Rightarrow N a = a(N-1)$

Let $\hat{N}|\lambda\rangle = \lambda|\lambda\rangle$ be eigenstates of $\hat{N}$ (normalized)

Then $\hat{N}(a^\dagger|\lambda\rangle) = (\lambda+1)(a^\dagger|\lambda\rangle)$ ie $a^\dagger|\lambda\rangle \sim |\lambda+1\rangle$ raising

$\hat{N}(a|\lambda\rangle) = (\lambda-1)(a|\lambda\rangle)$ ie $a|\lambda\rangle \sim |\lambda-1\rangle$ lowering



$|\lambda\rangle$ are also energy eigenstates

$$\hat{H}|\lambda\rangle = \underbrace{\hbar\omega(\lambda+\tfrac{1}{2})}_{E}|\lambda\rangle$$

From analysis of Schrodinger eqn $\Rightarrow E > V_{min} = 0$

$$\Rightarrow \lambda > -\tfrac{1}{2}$$

$\exists\, \lambda_{min} \ni \quad a|\lambda_{min}\rangle = 0$

$$\langle\lambda_{min}|a^{\dagger} = 0$$

$$0 = \langle\lambda_{min}|a^{\dagger}a|\lambda_{min}\rangle = \langle\lambda_{min}|N|\lambda_{min}\rangle = \lambda_{min}$$

$E_{min} = \hbar\omega(\lambda_{min}+\tfrac{1}{2}) = \tfrac{1}{2}\hbar\omega$ ← ground st energy of harmonic oscillator

$|0\rangle$ = ground state

All other states obtained by raising λ by 1 $\Rightarrow |n\rangle$ n = non-neg integers

Eigenstate $|n\rangle$ has energy $E = \hbar\omega(n+\tfrac{1}{2})$

one can show: $a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$
$a|n\rangle = \sqrt{n}\,|n-1\rangle$

Mnemonic = $\sqrt{\text{larger}}$

$|1\rangle = a^{\dagger}|0\rangle$

$|2\rangle = \frac{1}{\sqrt{2}}a^{\dagger}|1\rangle$

$|3\rangle = \frac{1}{\sqrt{3}}a^{\dagger}|2\rangle$

$$\Rightarrow \boxed{|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger})^{n}|0\rangle}$$

Harmonic oscillator in position space

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\, x + \sqrt{\frac{\hbar}{2m\omega}}\, \frac{d}{dx}$$

$$= \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) \qquad y = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\therefore \hat{a}^{+} = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$$

Ground state obeys $\quad a|0\rangle = 0$

In position space $\quad \left(y + \frac{d}{dy}\right) u_0(y) = 0$

$$\frac{du_0}{dy} = -y\, u_0$$

$$\frac{du_0}{u_0} = -y\, dy$$

$$\ln u_0 = -\frac{1}{2}y^2 + \text{const}$$

$$u_0 = C e^{-\frac{1}{2}y^2}$$

$$= C e^{-\frac{m\omega x^2}{2\hbar}}$$

Normalize: $\quad u_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^{+})^n |0\rangle$$

In position space

$$u_n = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}} \left[y - \frac{d}{dy} \right] \right)^n \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{1}{2}y^2}$$

$$= \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{n!\,2^n}} \underbrace{\left(y - \frac{d}{dy}\right)^n e^{-\frac{1}{2}y^2}}_{\text{this is } H_n(y)\, e^{-\frac{1}{2}y^2}}$$

e.g. $\left(y - \frac{d}{dy}\right) e^{-\frac{1}{2}y^2}$

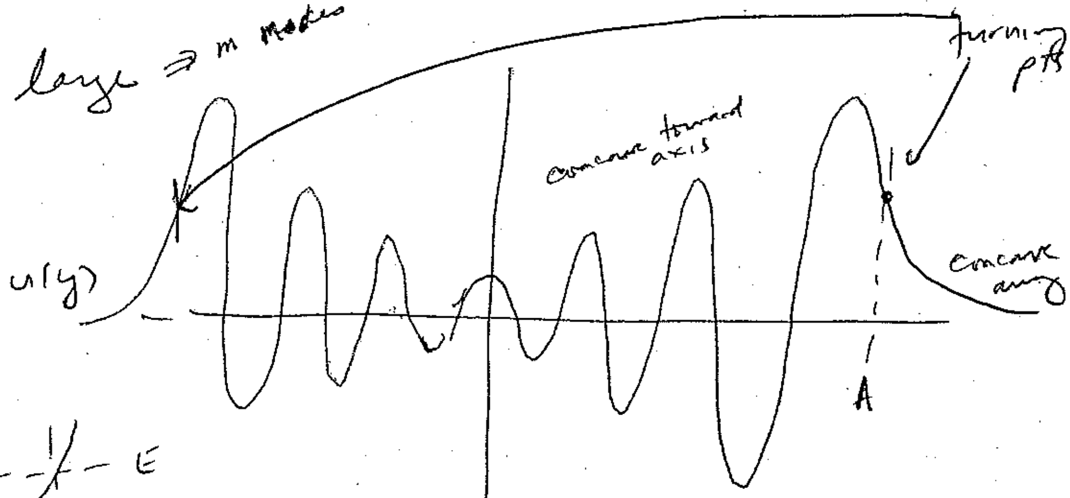
$$= [y - (-y)]\, e^{-\frac{1}{2}y^2} = 2y\, e^{-\frac{1}{2}y^2}$$

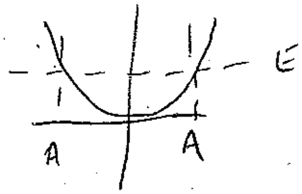
$$H_1(y) = 2y$$

n	$H_n(y)$
0	1
1	$2y$
2	$4y^2 - 2$
3	$8y^3 + 2y$
4	$16y^4 - 48y^2 + 12$

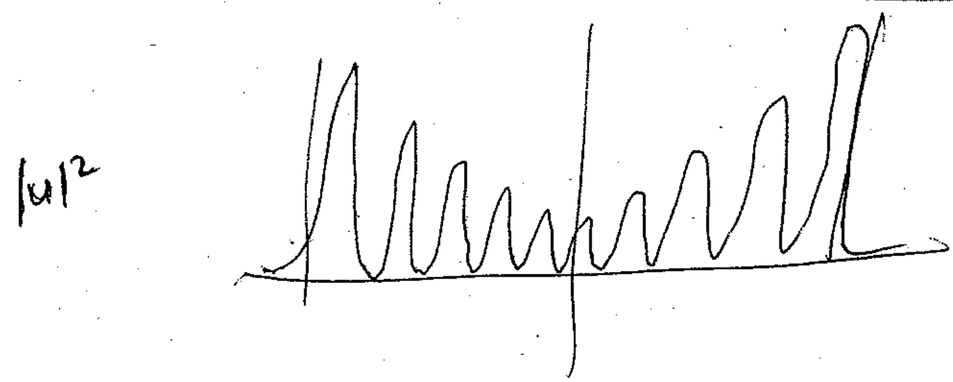
$$u_n = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{n!\,2^n}} H_n(y)\, e^{-\frac{1}{2}y^2}$$

• m large ⇒ m nodes

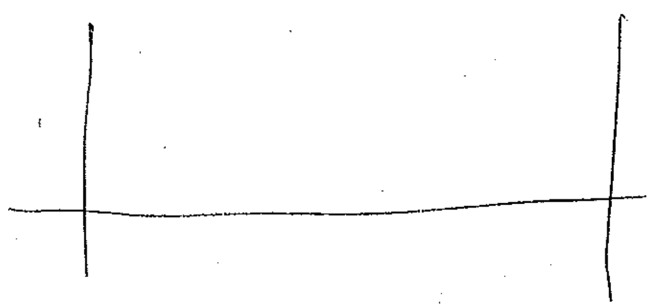




~~wtta~~ beyond turning points, $|u|^2 \neq 0$



Why is it more probable to find ~~a~~ particle near turning points? Because particle moves ~~slow~~ more slowly there!



~~Skip~~

$\frac{\Delta x}{\Delta x}$ 10

Prob ~ time spent ~ $\frac{1}{v}$

$\left[\begin{array}{l}\text{amt of time spent} \\ \text{btw } x \text{ \& } x+\Delta x \\ \quad x \quad x+\Delta x\end{array} \quad \Delta t = \frac{\Delta x}{v}\right]$

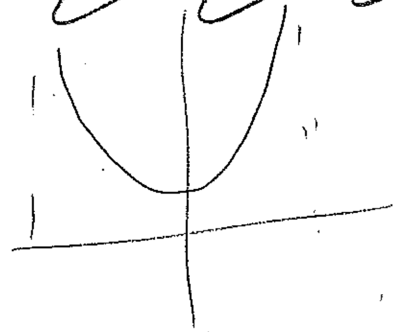
33 pgs

33-7

$\cancel{x = A\cos\omega t}$

$\cancel{v = -\omega A\sin\omega t = -\omega\sqrt{A^2 - A^2\cos^2\omega t} = -\omega\sqrt{A^2-x^2}}$

$P(x) \sim \frac{1}{v} \sim \frac{1}{\sqrt{A^2-x^2}}$

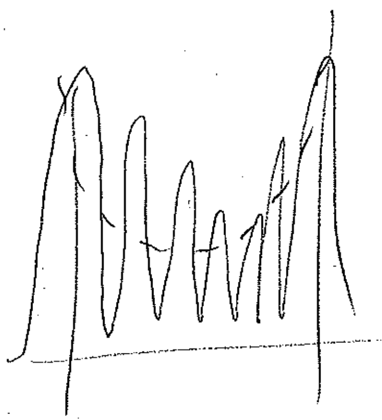


$\frac{1}{2}mv^2 = E - V(x) = \frac{1}{2}m\omega^2(A^2-x^2)$

CM $\quad v = \pm\,\omega\sqrt{A^2-x^2}$

do a problem

QM



correspondence principle

QM → CM as $n \to \infty$

The power of the operator approach

33-5

$$\langle n | X^2 | n \rangle = \int dx \; x^2 |u_n(x)|^2$$

$$\sim \int dy \; y^2 H_n^2(y) e^{-y^2}$$

difficult for $n = 143$!

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$\langle n | X^2 | n \rangle = \frac{\hbar}{2m\omega} \underbrace{\langle n | (a + a^\dagger)(a + a^\dagger) | n \rangle}$$

$$\langle n | a\,\underbrace{a | n \rangle}_{\sqrt{n}\,|n-1\rangle} + \langle n | a\,\underbrace{a^\dagger | n \rangle}_{\sqrt{n+1}\,|n\rangle} + \langle n | a^\dagger \underbrace{a | n \rangle}_{\sqrt{n}\,|n-1\rangle} + \langle n | a^\dagger a^\dagger | n \rangle$$

$$\underbrace{\underbrace{\sqrt{n}\sqrt{n-1}\,|n-2\rangle}}_{0} \qquad \underbrace{\underbrace{(n+1)\,|n\rangle}}_{n+1} \qquad \underbrace{\underbrace{n\,|n\rangle}}_{n}$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

In 2010, I skipped the following but talked about quiz of some [illegible]

→ ~~phonons~~

→ zero pt energy

→ phonons

(also terms: $\langle a \rangle$ ~~[illegible]~~ $a^2 = (a^\dagger)^2 = 0$

[Problem: show $\langle n | X | n \rangle = 0$

$\langle n | P | n \rangle = 0$]