Time evolution of a Free particle in 1 dimension

$$E = K = \tfrac{1}{2}mv^2 = \frac{p^2}{2m} \qquad \text{(all energy is kinetic)}$$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle$$

To solve, expand in energy eigenstates.
but for a free particle, momentum eigenstates are energy eigenstates
Easiest to solve free particle evolution in momentum basis

$$|\psi(t)\rangle = \int dp\, \phi(p,t)\,|p\rangle = e^{-\frac{iHt}{\hbar}}\int dp\, \phi(p,0)\,|p\rangle$$

$$= \int dp\, \phi(p,0)\, e^{-\frac{ip^2t}{2m\hbar}}\,|p\rangle$$

$$\text{so} \quad \phi(p,t) = \phi(p,0)\, e^{-\frac{ip^2t}{2m\hbar}}$$

$$|\phi(p,t)|^2 = |\phi(p,0)|^2 \Rightarrow \left.\begin{array}{c}\text{momentum}\\ \text{probability}\\ \text{density}\end{array}\right\} \text{doesn't change}$$

because $[p, H] = 0$

[cf www.eng.fsu.edu/~dommelen to demo of QM vs phase velocity]

In position space

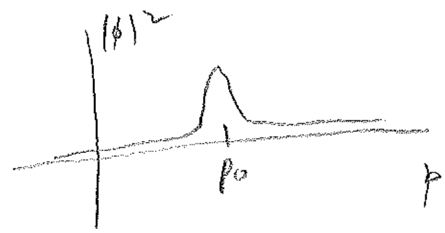
$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int dp\, \phi(p,t)\, e^{\frac{ipx}{\hbar}}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int dp\, \phi(p,0)\, \underbrace{e^{i\left(\frac{px}{\hbar} - \frac{p^2t}{2m\hbar}\right)}}_{\text{traveling wave}}}_{\substack{\text{linear combination of travelling waves}\\ = \text{"wave packet"}}}$$

Speed of wave packet = group velocity = $\frac{d\omega}{dk} = \frac{p}{m} = v_{classical}$

Consider a gaussian momentum distribution

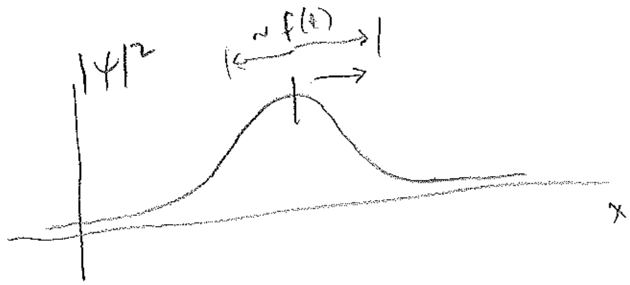
$$\phi(p,0) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} e^{-\beta(p-p_0)^2}$$



Position space wavefunction evolves to some $\Psi(x,t)$ ~~that~~ such that

$$|\Psi(x,t)|^2 = \left(\frac{1}{2\pi f^2(t)}\right)^{\frac{1}{2}} e^{-\frac{(x-\frac{p_0}{m}t)^2}{2f^2(t)}}$$ [HW]

You will calc $f(t)$ in HW, but $f(t)$ grows monotonically w/ t



$\langle x \rangle = \frac{p_0}{m} t$

Peak moves w/ speed $\frac{p_0}{m}$. $\Rightarrow$ not a stationary state because of interference bw energy eigenstates

gaussian spreads out w/ time (position becomes more uncertain)

[HW: ~~show~~ derive $\frac{d}{dt}\langle x \rangle = \frac{\langle p \rangle}{m}$; show that it is satisfied here.]

QM. w/ Mathematica