

[Besides being a generator of translations]

Momentum is an observable

$$\hat{p} |p\rangle = p |p\rangle$$

p = possible result of a measurement of momentum = any real #

$|p\rangle$ = state with perfectly well-defined momentum (idealization)

Completeness

$$\int dp |p\rangle \langle p| = 1$$

orthonormality

$$\langle p' | p \rangle = \delta(p - p')$$

$$|p\rangle = \int dx |x\rangle \underbrace{\langle x|p\rangle}_{\equiv \psi_p(x)} = \text{wave-fn. represented in pos. space} \\ (\text{i.e. in } x\text{-basis})$$

$$P|p\rangle = p|p\rangle$$

in
pos. space

$$\frac{\hbar}{i} \frac{d}{dx} \psi_p(x) = p \psi_p$$

$$\frac{d\psi_p}{\psi_p} = \frac{ip}{\hbar} dx$$

$$\ln \psi_p = \frac{ipx}{\hbar} + \text{const.}$$

$$\psi_p(x) = A e^{\frac{ipx}{\hbar}}$$

$$\Rightarrow \langle x|p\rangle = A e^{\frac{ipx}{\hbar}} \quad (A \text{ still to be determined})$$

NB. $|p\rangle$ not normalizable: $\int_{-\infty}^{\infty} dx |\psi_p(x)|^2 = \int_{-\infty}^{\infty} dx |A|^2 = \infty$.

Also, equally likely to be anywhere in space!

$$\Delta x \geq \frac{\hbar}{2\Delta p} = \infty$$

$|p\rangle$ form an orthonormal basis in the Dirac sense

$$\text{ie, } \langle p'|p\rangle = \delta(p'-p)$$

Use this to determine A .

$$\delta(p-p') = \int_{-\infty}^{\infty} dx \langle p'|x\rangle \langle x|p\rangle$$

$$= \int_{-\infty}^{\infty} dx A^* e^{-\frac{ip'x}{\hbar}} A e^{\frac{ipx}{\hbar}}$$

$$= |A|^2 \int_{-\infty}^{\infty} dx e^{\frac{i(p-p')x}{\hbar}}$$

[Can "see" that r.h.s = 0 if $p \neq p'$ because $\cos + i\sin = \text{area variable}$.
But need to choose particular $|A|^2$ for it to be normalized

$$= |A|^2 \lim_{L \rightarrow \infty} \underbrace{\int_{-L}^L dx e^{\frac{i(p-p')x}{\hbar}}}_{\frac{\hbar}{i(p-p')} \underbrace{e^{\frac{i(p-p')x}{\hbar}} \Big|_{-L}^L}_{2i \sin \left(\frac{(p-p')L}{\hbar} \right)}}$$

$$\delta(p-p') = 2\hbar |A|^2 \lim_{L \rightarrow \infty} \frac{\sin \left(\frac{(p-p')L}{\hbar} \right)}{(p-p')}$$

Integrate both sides wrt. p

$$\int_{-\infty}^{\infty} \delta(p-p') dp = 2\pi\hbar |A|^2 \lim_{L \rightarrow \infty} \underbrace{\int_{-\infty}^{\infty} dp \frac{1}{(p-p')} \sin\left[\frac{(p-p')L}{\hbar}\right]}$$

$$\text{let } y = \frac{(p-p')L}{\hbar}$$

$$\underbrace{\int_{-\infty}^{\infty} dy \frac{\sin y}{y}}$$

π (see next pr)

$$1 = 2\pi\hbar |A|^2$$

$$A = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\Rightarrow \delta(p-p') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx e^{\frac{i(p-p')x}{\hbar}}$$

$$\Rightarrow \boxed{\langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}}$$

$$|p\rangle = \int dx |x\rangle \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx e^{ipx/\hbar} |x\rangle$$

$$|x\rangle = \int dp |p\rangle \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp e^{-ipx/\hbar} |p\rangle$$

[change of basis]

↓
Unitary transformation

$$U = e^{-\frac{icap}{\hbar}}$$

$$U|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp e^{-ipx/\hbar} e^{-\frac{icap}{\hbar}} |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp e^{-\frac{ip(x+a)}{\hbar}} |p\rangle = |x+a\rangle$$