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Classical $\left\{ \begin{array}{l} \text{Inv't of l.p under spatial transl} \Rightarrow \text{cons } p \\ \quad \text{" " " time} \Rightarrow \text{cons } E \\ \quad \text{... rotation} \Rightarrow \text{cons } \vec{J} \end{array} \right.$

Momentum operator

Recall: angular momentum operator $\vec{J}$ generates rotations

ie rotations through $\phi\hat{n}$ are implemented by $U = e^{-\frac{i\vec{J}\cdot\hat{n}\phi}{\hbar}}$

Recall Hamiltonian H generates time translations

ie time evolution by t is implemented by $U = e^{-\frac{iHt}{\hbar}}$

Claim: momentum operator P generates spatial translations

ie translation by a is implemented by $U = e^{-\frac{iPa}{\hbar}}$

, e.g. $U|x\rangle = |x+a\rangle$

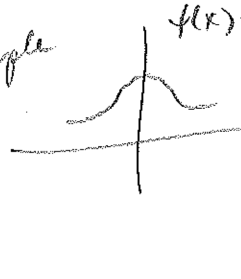
Then $U|\psi\rangle = |\psi'\rangle$ ~~where $\psi'(x) = \psi(x-a)$~~

$$\cancel{|\psi'\rangle = \int dx\, \psi'(x)|x\rangle}$$

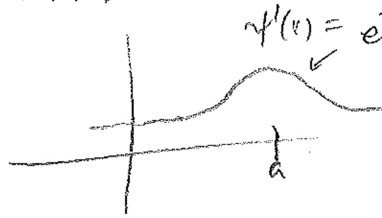
$$\begin{aligned} U|\psi\rangle &= \int dx\, \psi(x)\, U|x\rangle \\ &= \int dx\, \psi(x)\, |x+a\rangle \\ y=x+a \quad &= \int dy\, \psi(y-a)\,|y\rangle \\ \text{rename} \quad &= \int dx\, \psi(x-a)\,|x\rangle \end{aligned} \quad \rightarrow \psi'(x) = \psi(x-a)$$

$$\text{true } |\psi'\rangle = \int dx\, \psi'(x)|x\rangle$$

for example $\psi(x) = e^{-\alpha x^2}$



$\psi'(x) = e^{-\alpha(x-a)^2} = \psi(x-a)$



In position space, U acts on wavefunction as

$$U_{pos}\psi(x) = \psi(x-a)$$

$$= \psi(x) - a\frac{d\psi}{dx} + \frac{1}{2}a^2\frac{d^2\psi}{dx^2} - \frac{1}{3!}a^3\frac{d^3\psi}{dx^3}$$

$$e^{-\frac{ia\hat{P}_{pos}}{\hbar}}\psi(x) = e^{-a\frac{d}{dx}}\psi(x)$$

$\therefore$ in position space: $\quad \hat{P}_{pos.} = \frac{\hbar}{i}\frac{d}{dx}$

$\hat{X}$ acts on $\psi(x)$ to give $x\,\psi(x)$

$\hat{p}$ acts on $\psi(x)$ to give $\frac{\hbar}{i}\frac{d}{dx}\psi(x)$

both are linear operators

Consider $[\hat{X},\hat{P}]\,\psi(x) = XP\psi - PX\psi$

$$= x\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}\underbrace{\frac{d}{dx}(x\psi)}_{\psi + x\frac{d\psi}{dx}}$$

$$= i\hbar\,\psi(x)$$

Since this is true for arbitrary $\psi(x)$, we find

$$\boxed{[\hat{X},\hat{P}] = i\hbar\,\mathbb{1}}$$ Heisenberg commutation relations

X, P incompatible.

Recall generalized uncertainty relation

$$\Delta A\,\Delta B \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$$

$$\Rightarrow \Delta x\,\Delta p \geq \frac{\hbar}{2}$$

X, P incompatible $\Rightarrow$ eigenstates of x are not eigenstates of p.

Problem: compute $[X, P^n]$
$[X^n, P]$
Show $[V(X), P] = i\hbar V'(x)$