

Position observable

Review:

observable $A \rightarrow$ hermitian operator $\hat{A}$

eigenvalue eqn $\hat{A} |a_n\rangle = a_n |a_n\rangle$

eigenvalues: results of possible measurements of A
 eigenstates: state of determinate A

orthonormality $\langle a_n | a_m \rangle = \delta_{nm}$

completeness $\sum_n |a_n\rangle \langle a_n| = 1$

decomposition of an arbitrary state into eigenstates

$$|\psi\rangle = \sum |a_n\rangle \langle a_n | \psi \rangle = \sum \psi_n |a_n\rangle$$

$\psi_n = \langle a_n | \psi \rangle =$ prob. amplitudes

How does $\hat{A}$ act on $|\psi\rangle$?

$$\hat{A} |\psi\rangle = \sum \hat{A} |a_n\rangle \langle a_n | \psi \rangle = \sum a_n \psi_n |a_n\rangle$$

$|\psi\rangle$ is normalized

$$\Rightarrow 1 = \langle \psi | \psi \rangle = \sum \langle \psi | a_n \rangle \langle a_n | \psi \rangle = \sum \psi_n^* \psi_n = \sum |\psi_n|^2$$

$$\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum a_n \psi_n \underbrace{\langle \psi | a_n \rangle}_{\psi_n^*} = \sum a_n |\psi_n|^2$$

$\Rightarrow |\psi_n|^2 =$ prob. of measuring a_n

position operator $\hat{x}$

$$\hat{x}|x\rangle = x|x\rangle$$

value x = possible result of a measurement of position

= any real number
spectrum of $\hat{x}$ is $\mathbb{R}$

$|x\rangle$ = state of perfectly well-defined position (idealization)
unphysical

Completeness (since all values of x are allowed, $\Sigma \rightarrow \int$)

$$\int dx |x\rangle \langle x| = \mathbb{I}$$

Then $|\psi\rangle = \int dx |x\rangle \langle x|\psi\rangle = \int dx \psi(x) |x\rangle$

state of indeterminate position

where $\psi(x) = \langle x|\psi\rangle$ = a number that depends on ψ and x
ie a function of x !

= probability amplitude

= position-space wavefunction

[How to think of $\psi(x)$ as a vector: $\int dx \psi(x) |x\rangle \sim \sum_{n=-\infty}^{\infty} \psi(n) |x=n\rangle \Rightarrow \begin{pmatrix} \psi(-1) \\ \psi(0) \\ \psi(1) \\ \vdots \end{pmatrix}$

Suppose $|\psi\rangle$ is normalized

$$\begin{aligned} 1 &= \langle \psi | \psi \rangle = \int dx \langle \psi | x \rangle \langle x | \psi \rangle \\ &= \int dx \psi^*(x) \psi(x) \\ &= \int dx |\psi(x)|^2 \end{aligned}$$

$|\psi(x)|^2 =$ probability density of obtaining x

$$\left[\sum_n \psi_n^* \psi(n) \right] = \begin{pmatrix} \psi_1^* & \psi_2^* & \dots \end{pmatrix} \begin{pmatrix} \psi(1) \\ \psi(2) \\ \vdots \end{pmatrix}$$

A function that obeys $\int dx |\psi(x)|^2 < \infty$ called
"square integrable" or "normalizable"

$\psi(x) \xrightarrow{x \rightarrow \pm\infty} 0$ faster than $\frac{1}{\sqrt{|x|}}$

How does $\hat{X}$ act on a state $|\psi\rangle$?

$$\text{Let } |\psi\rangle = \int dx \psi(x) |x\rangle$$

$$\text{Then } \hat{X}|\psi\rangle = \int dx \psi(x) \hat{X}|x\rangle = \int dx x \psi(x) |x\rangle$$

$$\text{so if } \overset{\text{abstract}}{|\psi\rangle} \rightarrow \overset{\text{position space}}{\psi(x)}$$

$$\text{then } \hat{X}|\psi\rangle \rightarrow x \psi(x) \quad (\text{multiply } \psi \text{ by } x)$$

$$\text{ie } \hat{X} \rightarrow (\hat{X})_{\text{pos}} = x$$

Expectation value

$$\langle x \rangle = \langle \psi | \hat{X} | \psi \rangle$$

$$= \langle \psi | \int dx x \psi(x) |x\rangle$$

$$= \int dx \underbrace{\langle \psi | x \rangle}_{\psi^*(x)} x \psi(x)$$

$$= \int dx \psi^*(x) x \psi(x)$$

$$= \int dx x |\psi(x)|^2$$

prob. density

[as in 2140]

$$\text{Fn self: } \hat{X} = \int dx dx' |x\rangle \underbrace{\langle x | x' \rangle}_{x \delta(x-x')} \langle x' |$$

$$= \int dx |x\rangle x \langle x|$$

ie diagonal

What about orthonormality?

What is $\langle x' | x \rangle$? Expect = 0 if $x \neq x'$

Expect = 1? if $x = x'$
no!

Because x is a continuous variable $\langle x | x \rangle = \infty$

but ∞ is a subtle concept so let's be more careful

Consider an observable A w/ discrete eigenvalues a_n

$$|\psi\rangle = \sum \psi_n |a_n\rangle$$

$$\psi_n = \langle a_n | \psi \rangle = \sum_m \langle a_n | a_m \rangle \langle a_m | \psi \rangle = \sum_m \delta_{nm} \psi_m \quad \checkmark$$

↑ Kronecker delta

Consider an observable X w/ continuous eigenvalues x

$$|\psi\rangle = \int dx' \psi(x') |x'\rangle$$

$$\psi(x') = \langle x' | \psi \rangle = \int dx \underbrace{\langle x' | x \rangle}_{\delta(x-x')} \psi(x)$$

$$\delta(x-x')$$

↑ Dirac delta "fun"

(continuum analog of δ_{mn})

What is this Dirac delta "function" $\delta(x)$?

$\delta(x)$ is defined by two properties:

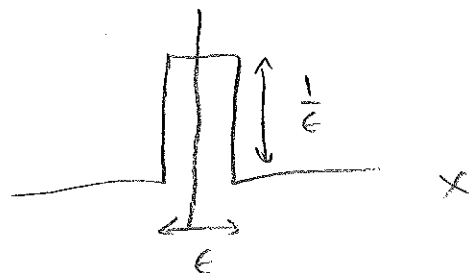
① $\delta(x) = 0$ if $x \neq 0$

② $\int_{-\infty}^{\infty} dx \delta(x) = 1$

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$\delta(x)$ is usually represented as a limit of a family of functions, for example

$$\delta(x) = \lim_{\epsilon \rightarrow 0} R_{\epsilon}(x) \quad \text{where} \quad R_{\epsilon}(x) = \begin{cases} \frac{1}{\epsilon}, & |x| < \frac{1}{2}\epsilon \\ 0, & |x| > \frac{1}{2}\epsilon \end{cases}$$



checks:

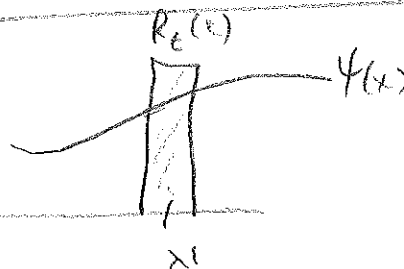
① for $x = a \neq 0$, then $R_{\epsilon}(a) = 0$ provided $\epsilon < 2|a|$.

Since $\epsilon \rightarrow 0$ is smaller than any $a \neq 0$, $\delta(a) = 0$

② $\int_{-\infty}^{\infty} R_{\epsilon}(x) dx = \frac{1}{\epsilon} \cdot \epsilon = 1$ for any ϵ , so $\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} R_{\epsilon}(x) dx = 1$

Consider

$$\int_{-\infty}^{\infty} R_{\epsilon}(x-x') \psi(x) dx =$$



$$= \frac{1}{\epsilon} \int_{x' - \frac{\epsilon}{2}}^{x' + \frac{\epsilon}{2}} \psi(x) dx \approx \psi(x') \cdot \frac{1}{\epsilon} \int dx = \psi(x')$$

↑
for small ϵ

$$\therefore \int_{-\infty}^{\infty} \delta(x-x') \psi(x) dx = \psi(x')$$

But this is precisely what we need for $\langle x' | \psi \rangle$

$$\boxed{\langle x' | \psi \rangle = \int_{-\infty}^{\infty} \delta(x-x') \psi(x) dx}$$

To prove: $\delta(x)$ is an even function

need to establish that $\delta(-x)$ obeys the defining properties

$$\textcircled{1} \quad \delta(-x) = 0 \quad \text{if } x \neq 0 \quad \checkmark$$

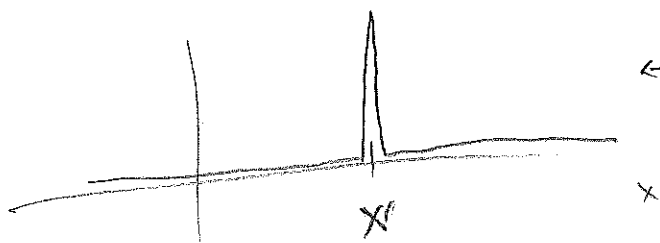
$$\textcircled{2} \quad \int_{x=-\infty}^{x=\infty} dx \delta(-x) = \int_{y=-\infty}^{y=\infty} (-dy) \delta(y) = \int_{y=-\infty}^{y=\infty} dy \delta(y) = 1 \quad \checkmark$$

$y = -x$
 $dy = -dx$

$$\therefore \langle x' | x \rangle = \delta(x - x') = \delta(x' - x) = \langle x | x' \rangle$$

$\langle x | \psi \rangle = \psi(x)$ is position space wavefunction of $|\psi\rangle$

$\Rightarrow \langle x | x' \rangle = \delta(x - x')$ is the position space wavefunction of $|x'\rangle$



← particle is perfectly localized

But $\delta(x - x')$ is not square integrable!

$$\int_{-\infty}^{\infty} dx \delta(x - x') \delta(x - x') = \delta(0) = \infty!$$

$|x\rangle$ not normalizable