Precession of spin in a magnetic field

classical analysis

$E = -\vec{\mu}\cdot\vec{B}$

torque $\vec{\tau} = \vec{\mu}\times\vec{B}$

compass needle will swing back & forth + come to rest (due to friction) but spin will precess about mag field axis

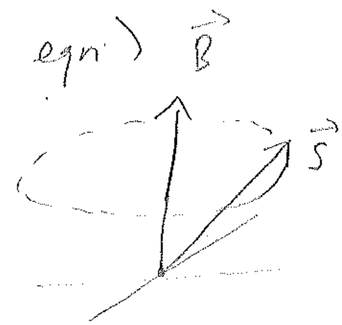
$\frac{d\vec{p}}{dt} = \vec{F}$; $\frac{d\vec{J}}{dt} = \vec{\tau}$ (classically)

Consider a free spin $\vec{J} = \vec{S}$

$\vec{\mu} = g\frac{q}{2m}\vec{S}$

electron: $q = -e$, $g \simeq 2$
proton: $q = +e$, $g \cong 5.6$

$$\frac{d\vec{S}}{dt} = \frac{gq}{2m}\vec{S}\times\vec{B} \quad \text{(gyroscope eqn)}$$



Let $\vec{B} = B\hat{z}$

$$\begin{cases} \frac{dS_z}{dt} = 0 \quad \Rightarrow S_z = \text{const} \\ \frac{dS_x}{dt} = \omega_L S_y \\ \frac{dS_y}{dt} = -\omega_L S_x \end{cases}$$

where $\omega_L = \frac{gqB}{2m}$ = Larmor frequency

$$\Rightarrow \frac{d^2S_x}{dt^2} + \omega_L^2 S_x = 0 \quad \Rightarrow \begin{cases} S_x = S_x(0)\cos(\omega_L t) \\ S_y = -S_x(0)\sin(\omega_L t) \\ S_z = S_z(0) \end{cases}$$

clockwise rotation

Larmor period $T = \frac{2\pi}{\omega_L}$

quantum analysis

$$E = -\vec{\mu}\cdot\vec{B} = -\left(\frac{gqB}{2m}\right)S_z = -\omega_L S_z$$

$$\Rightarrow \quad \hat{H} = -\omega_L \hat{S}_z$$

H, S_z compatible $\Rightarrow$ share eigenstates $\nearrow$ S_z conserved

$$H\,|\pm\rangle_z = \mp\frac{\hbar\omega_L}{2}|\pm\rangle_z \qquad \text{ie} \quad E_\pm = \mp\frac{\hbar\omega_L}{2}$$

Initial state

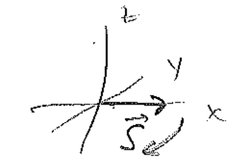
$$|\psi(0)\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle = \alpha e^{-\frac{iE_+t}{\hbar}}|+\rangle_z + \beta e^{-\frac{iE_-t}{\hbar}}|-\rangle_z$$

$$= \alpha e^{+\frac{i\omega_L t}{2}}|+\rangle_z + \beta e^{-\frac{i\omega_L t}{2}}|-\rangle_z$$

$$\text{Spin } \psi(t) = \begin{pmatrix} \alpha e^{\frac{i\omega_L t}{2}} \\ \beta e^{-\frac{i\omega_L t}{2}} \end{pmatrix} = e^{\frac{i\omega_L t}{2}}\begin{pmatrix} \alpha \\ \beta e^{-i\omega_L t} \end{pmatrix}$$

Suppose initial spin points along +x axis



$\downarrow$ expect precession clockwise around z-axis w/ period $T = \frac{2\pi}{\omega_L}$

$$|\psi(0)\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$$

Then $|\psi(t)\rangle = e^{\frac{i\omega_L t}{2}} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{-i\omega_L t}\end{pmatrix}$

After $\frac{1}{4}$ period: $\omega_L t = \frac{\omega_L T}{4} = \frac{\pi}{2}$ $\qquad |\psi(\tfrac{T}{4})\rangle = e^{\frac{i\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$e^{-i\omega_L t} = e^{-\frac{i\pi}{2}} = -i$ $\qquad \sim |-y\rangle$

After $\frac{1}{2}$ period $\omega_L t = \pi$ $\qquad |\psi(\tfrac{T}{2})\rangle = e^{\frac{i\pi}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \sim |-x\rangle$

$e^{-i\omega_L t} = -1$

After 1 period $\omega_L t = 2\pi$ $\qquad |\psi(T)\rangle = \underbrace{e^{i\pi}}_{-1}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \sim |+x\rangle$

$\uparrow$ can detect this using neutron interferometer

$$\boxed{e^{-\frac{iHt}{\hbar}} = e^{\frac{i\omega_L t}{\hbar}S_z} = e^{-\frac{i}{\hbar}\underbrace{(-\omega_L t)}_{\text{rotation angle about }\hat{z}} S_z}}$$

$\nwarrow$ time for 1 complete rotation

Problem: Calc $\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle$ via Heisenberg eqns.
Show same as classical eqns