2014



[if timing is bad, could move 15-1 up to here]

Time evolution of the state vector

Let the system be described by state $|\psi(0)\rangle$ at $t=0$.

Can we predict that state $|\psi(t)\rangle$ at time $t$?

Let $U(t)$ be a unitary operator that implements translations in time

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \quad \Rightarrow \quad \langle\psi(t)| = \langle\psi(0)|\, U^\dagger(t)$$

$U$ must be unitary so that probability is conserved

ie so that a state vector remains normalized

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\, \underbrace{U^\dagger U}_{\mathbb{I} \text{ if } U \text{ is unitary}} |\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

Generalize to $U(t_2, t_1)$ where $|\psi(t_2)\rangle = U(t_2,t_1)\,|\psi(t_1)\rangle$

Satisfies composition property

$$|\psi(t_3)\rangle = U(t_3,t_2)|\psi(t_2)\rangle = U(t_3 t_2)U(t_2 t_1)\,|\psi(t_1)\rangle$$

$$U(t_3,t_1) = U(t_3,t_2)\,U(t_2,t_1)$$

We need to find a unitary operator $U(t)$ that obeys the composition property, and satisfies initial condition $U(0) = \mathbb{I}$

In CM, invariance under time translations implies existence of conserved quantity, energy (Noether's theorem)

Claim: the operator associated w/ energy, the Hamiltonian $H$, generates time translations, i.e.

$$U(t_2,t_1) = e^{-\frac{iH}{\hbar}(t_2-t_1)} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{iH}{\hbar}(t_2-t_1)\right)^n \mathbb{1}$$

- Check: $U(t,t) = e^0 = \mathbb{1}$
- Check: satisfies composition

$$e^{-\frac{iH}{\hbar}(t_3-t_1)} = e^{-\frac{iH}{\hbar}(t_3-t_2)}\, e^{-\frac{iH}{\hbar}(t_2-t_1)}$$

Since $e^{\alpha A} e^{\beta B} = e^{\alpha A + \beta B}$ provided $[A,B]=0$

[Hw: prove this] $\quad e^{\alpha A} e^{\beta A} = e^{(\alpha+\beta)A}$

- Check: unitary

$$U^\dagger(t_2,t_1) = \sum \frac{1}{n!}\left(-\frac{iH}{\hbar}(t_2-t_1)\right)^{n\,\dagger}$$

$$= \sum \frac{1}{n!}\left(+\frac{iH^\dagger}{\hbar}(t_2-t_1)\right)^n$$

$$= \sum \frac{1}{n!}\left(+\frac{iH}{\hbar}(t_2-t_1)\right)^n \quad \text{by } H^\dagger = H$$

$$= e^{\frac{iH}{\hbar}(t_2-t_1)}$$

Then $U^\dagger(t_2,t_1)\,U(t_2,t_1) = e^{\frac{iH}{\hbar}(t_2-t_1)}\, e^{-\frac{iH}{\hbar}(t_2-t_1)} = e^0 = \mathbb{1}$

Claim: $U(t) = U(t,0)$ satisfies

$$\boxed{\frac{dU}{dt} = \frac{1}{i\hbar} H U}$$

Proof:
$$U(t+dt, 0) = U(t+dt, t)\, U(t,0)$$
$$= e^{-\frac{i}{\hbar} H\, dt}\; U(t,0)$$
$$\approx \left(1 - \frac{i}{\hbar} H\, dt\right) U(t,0)$$

$$U(t+dt) - U(t) = \left(\frac{1}{i\hbar} H\, dt\right) U(t)$$

$$\frac{dU}{dt} = \lim_{dt\to 0} \frac{U(t+dt) - U(t)}{dt} = \frac{1}{i\hbar} H\, U(t) \qquad \text{QED}$$

This implies the time dependent Schrodinger eqn.

$$\frac{d}{dt}|\psi(t)\rangle = \frac{d}{dt} U(t)\, |\psi(0)\rangle$$
$$= \frac{1}{i\hbar} H\, U(t)\, |\psi(0)\rangle$$
$$= \frac{1}{i\hbar} H\, |\psi(t)\rangle$$

$$\boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = H\, |\psi(t)\rangle}$$

Solving the Schrödinger eqn

easy if you know the energy eigenstates $H|E_n\rangle = E_n|E_n\rangle$

Expand the initial state in the energy basis

Recall $\mathbb{1} = \sum_n |E_n\rangle\langle E_n|$

$$|\psi(0)\rangle = \mathbb{1}\,|\psi(0)\rangle = \sum |E_n\rangle \underbrace{\langle E_n|\psi(0)\rangle}_{c_n(0)}$$ — coefficients in energy basis

$$U(t) = e^{-\frac{iHt}{\hbar}} = \sum_n e^{-\frac{iHt}{\hbar}}|E_n\rangle\langle E_n| = \sum e^{-\frac{iE_nt}{\hbar}}|E_n\rangle\langle E_n|$$

Then

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \sum_n e^{-\frac{iE_nt}{\hbar}}|E_n\rangle\langle E_n|\psi(0)\rangle$$

$$= \sum_n c_n(0)e^{-\frac{iE_nt}{\hbar}}|E_n\rangle$$

$$= \sum_n c_n(t)|E_n\rangle$$

∴ each eigenstate gets multiplied by a (different) phase

$$|\psi(t)\rangle = \underbrace{e^{-\frac{iE_1t}{\hbar}}}_{\substack{\text{overall phase} \\ \text{is physically irrelevant}}} \sum c_n(0)\underbrace{e^{-\frac{i(E_n-E_1)t}{\hbar}}}_{\text{relative phase}}|E_n\rangle$$

relative phase (which are physically relevant) only depend on energy differences (just as in C.M)

How does the expectation value of an observable A change in time?

$$\underbrace{\langle A \rangle}_{\substack{\text{change in}\\\text{time}\\\text{because}\\\text{state changes}}} = \langle \psi(t) | A | \psi(t) \rangle = \langle \psi(0) | U^\dagger(t) A U(t) | \psi(0) \rangle$$

↖ states carry time dependence

↖ operator generally not time dependent

$$\frac{d}{dt}\langle A \rangle = \langle \psi(0) | U^\dagger(t) A \underbrace{\frac{dU}{dt}}_{\frac{1}{i\hbar} H U(t)} | \psi(0) \rangle$$

$$+ \langle \psi(0) | \underbrace{\frac{dU^\dagger}{dt}}_{-\frac{1}{i\hbar} U^\dagger(t) H} A \, U(t) | \psi(0) \rangle$$

$$= \frac{1}{i\hbar} \langle \psi(t) | AH - HA | \psi(t) \rangle$$

$$\boxed{\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar} \langle \psi(t) | [A, H] | \psi(t) \rangle}$$ <u>Ehrenfest's theorem</u>

→ if $[A, H] = 0$ then $\langle A \rangle$ does not change over time

(prob 3.31 of 2e)

Suppose we have some operator $A$ that commutes w/ $H$

$$[A, H] = 0$$

Then $A$ also commutes w/ $U(t)$

Proof: $AH = HA$

$$AH^n = HAH^{n-1} = H^2AH^{n-2} = \dots = H^nA$$

$$U = \sum \frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^n \Rightarrow AU = UA$$

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Suppose an initial state $|\psi(0)\rangle$ has a well-defined value of $A$

$$A|\psi(0)\rangle = a|\psi(0)\rangle$$

Then if $[A,H]=0$ the state into which it evolves has the same well-defined value

$$A|\psi(t)\rangle = AU|\psi(0)\rangle = UA|\psi(0)\rangle = Ua|\psi(0)\rangle = a|\psi(t)\rangle$$

ie it provides a "good quantum number" to characterize the state (which does not change in time)

Since $[H,H]=0$, the energy is a good quantum no. (eg hydrogen)

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Even if $|\psi(0)\rangle$ does not have a well-defined value of $A$, its expectation value will not change in time if $[A,H]=0$ (due to Ehrenfest).