(Recall)

A spin-j representation of angular momentum is a vector space of dimension $2j+1$.

Call this space $(\underset{\sim}{2j+1})$

It is spanned by basis states $|j, m\rangle$ where $m = -j, \dots, j$
(eigenstates of J^2 and J_z)

Most general state is linear combination
$\sum c_m |j, m\rangle$

E.g. spin-$\frac{1}{2}$ rep, or $\underset{\sim}{2}$ is spanned by $|\frac{1}{2}, \frac{1}{2}\rangle$ + $|\frac{1}{2}, -\frac{1}{2}\rangle$

For brevity, we'll often omit j: $|\frac{1}{2}\rangle$ + $|-\frac{1}{2}\rangle$

or even more briefly $|+\rangle$ + $|-\rangle$

Recall (from prob-set) $J_+ |j, m\rangle = \hbar \sqrt{(j-m)(j+m+1)}\ |j, m+1\rangle$
$J_- |j, m\rangle = \hbar \sqrt{(j+m)(j-m+1)}\ |j, m-1\rangle$

or ~~next~~ until turned in

Consider a system of two spins: w/ spin j_1 and spin j_2

e.g. proton + electron in grnd state of hydrogen

Basis states of the system are written

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle, \quad \text{or} \quad |m_1\rangle \otimes |m_2\rangle, \quad \text{or} \quad |m_1 ; m_2\rangle$$

N.B.

(j_1 + j_2 are implied)

e.g. two spin-$\frac{1}{2}$ particles, both spin up $\rightarrow$ $|\frac{1}{2} ; \frac{1}{2}\rangle$

" " down $\rightarrow$ $|-\frac{1}{2} ; -\frac{1}{2}\rangle$

also $|\frac{1}{2} ; -\frac{1}{2}\rangle$

$|-\frac{1}{2} ; \frac{1}{2}\rangle$

This space, called $\underbrace{\underset{\sim}{2} \otimes \underset{\sim}{2}}_{\text{(tensor) product space}}$ is 4-dimensional

more generally, $(2\underset{\sim}{j_1}+1) \otimes (2\underset{\sim}{j_2}+1)$ is $(2j_1+1)(2j_2+1)$ dim'l

The most general state of the system is a linear combination $\displaystyle\sum_{m_1, m_2} c_{m_1 m_2} |m_1 ; m_2\rangle$

Electron in hydrogen atom

Old quantum theory: described by 4 quantum numbers (Pauli, 1924)

principal $n = 1, 2, 3, \ldots$ $\Rightarrow$ determines energy $E_n = \frac{(-13.6\,eV)}{n^2}$

orbital $l = 0, 1, \ldots, n-1$ $\Rightarrow |\vec{L}| = l\hbar$

(magnetic) $m = -l, -l+1, \ldots, l$ $\Rightarrow L_z = m\hbar$

$s = \frac{1}{2}$ $\Rightarrow |\vec{S}| = \frac{1}{2}\hbar$

Spin $m_s = \pm\frac{1}{2}$ $\Rightarrow S_z = m_s\hbar$

Quantum mechanics

$|l, m\rangle$ describe states of orbital angular momentum

$|\frac{1}{2}, m_s\rangle$ describe states of spin of electron (intrinsic)

Therefore, apart from the principal quantum number states of an electron are described by the tensor products

$|l, m\rangle \otimes |\frac{1}{2}, m_s\rangle$ which belong to $\underset{\sim}{2l+1} \otimes \underset{\sim}{2}$

n	$l=0$	$l=1$	$l=2$	
$n=4$	$1 \otimes 2$ (4s)	$3 \otimes 2$ (4p)	$5 \otimes 2$ (4d)	$7 \otimes 2$ (4f)
$n=3$	$1 \otimes 2$ (3s)	$3 \otimes 2$ (3p)	$5 \otimes 2$ (3d)	
$n=2$	$1 \otimes 2$ (2s)	$3 \otimes 2$ (2p)		
$n=1$	$1 \otimes \underset{\sim}{2}$ (1s)			

Exclusion principle $\Rightarrow$ periodic table

An operator on the (tensor) product space is of the form $A \otimes B$ where A acts on 1st spin & B on the 2nd

$$A \otimes B \, |m_1; m_2\rangle = A \otimes B \, |m_1\rangle \otimes |m_2\rangle \equiv A|m_1\rangle \otimes B|m_2\rangle$$

Eg. spin of 1st particle $\vec{J}_1$ is $\vec{J}_1 \otimes 1$

" . . . 2nd particle $\vec{J}_2$ " $\mathbb{1} \otimes \vec{J}_2$

Eg. $J_{1z} \, |\tfrac{1}{2}; -\tfrac{1}{2}\rangle = \tfrac{1}{2}\hbar \cdot 1 \, |\tfrac{1}{2}; -\tfrac{1}{2}\rangle$

$J_{2z} \, |\tfrac{1}{2}; -\tfrac{1}{2}\rangle = 1 \cdot (-\tfrac{\hbar}{2}) \, |\tfrac{1}{2}; -\tfrac{1}{2}\rangle$

so $|m_1; m_2\rangle$ are eigenstates of J_1^2, J_{1z}, J_2^2, J_{2z}

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Product of operators ~~$(A_1 \otimes B_1) \cdot (A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$~~

Note that $\vec{J}_1$ and $\vec{J}_2$ <u>commute</u> w/ one another $\Rightarrow$ compatible

$$(\vec{J}_1 \otimes 1)(1 \otimes \vec{J}_2) = (\vec{J}_1 \otimes \vec{J}_2)$$

$$(1 \otimes \vec{J}_2)(\vec{J}_1 \otimes \mathbb{1}) = \vec{J}_1 \otimes \vec{J}_2 \quad \checkmark$$

ie (any) component of $\vec{J}_1$ commutes w/ any component of $\vec{J}_2$

Total angular momentum

Classically, total spin of the system is just $\vec{J} = \vec{J}_1 + \vec{J}_2$

QM, total spin is an operator $\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$

Act by J_z on the ~~tens~~ product space basis of $\underset{\sim}{2} \otimes \underset{\sim}{2}$

$$J_z \left| \tfrac{1}{2}; \tfrac{1}{2} \right\rangle = \tfrac{\hbar}{2} \left| \tfrac{1}{2}; \tfrac{1}{2} \right\rangle + \tfrac{\hbar}{2} \left| \tfrac{1}{2}; \tfrac{1}{2} \right\rangle = \hbar \left| \tfrac{1}{2}; \tfrac{1}{2} \right\rangle$$

$$J_z \left| -\tfrac{1}{2}; -\tfrac{1}{2} \right\rangle = -\hbar \left| -\tfrac{1}{2}; -\tfrac{1}{2} \right\rangle$$

$$J_z \left| \tfrac{1}{2}; -\tfrac{1}{2} \right\rangle = \tfrac{\hbar}{2} \left| \tfrac{1}{2}; \tfrac{1}{2} \right\rangle - \tfrac{\hbar}{2} \left| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle = 0$$

$$J_z \left| -\tfrac{1}{2}; \tfrac{1}{2} \right\rangle = 0$$

What rep. of total spin is this?

spin 1 and spin 0

$$\underset{\#1}{(\text{spin } \tfrac{1}{2})} \otimes \underset{\#2}{(\text{spin } \tfrac{1}{2})} = (\text{spin } 1) \oplus (\text{spin } 0)$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$$

(tensor) product space $\qquad$ direct sum

(Tensor) product space of individual spins = Direct sum space of total spin of system

(spin $\frac{1}{2}$) $\otimes$ (spin $\frac{1}{2}$) = (spin 1) $\oplus$ (spin 0)

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$$

$\vert m_1; m_2\rangle$	$m = m_1 + m_2$	$\vert j, m\rangle$
$\vert\frac{1}{2}; \frac{1}{2}\rangle$	=	$\vert 1,1\rangle$
$\vert+\frac{1}{2}; -\frac{1}{2}\rangle$ $\quad \vert-\frac{1}{2}; +\frac{1}{2}\rangle$	$\xleftrightarrow[\text{combinations}]{\text{linear}}$	$\vert 1,0\rangle$ $\quad \vert 0,0\rangle$
$\vert-\frac{1}{2}; -\frac{1}{2}\rangle$	=	$\vert 1,-1\rangle$

These are eigenstates of J_{1z} and J_{2z} (and $J_1^2 + J_2^2$)

These are also eigenstates of J_z because $J_z = J_{1z} + J_{2z}$

but they are not (necessarily) eigenstates of J^2

(because $[J^2, J_{1z}] \neq 0$
$[J^2, J_{2z}] \neq 0$)

These are eigenstates of J^2 and J_z

They are not (necessarily) eigenstates of J_{1z} and J_{2z}

our goal is to discover which linear combinations

i.e. find linear transf. between basis states

The only states on each side of $m = m_1 + m_2 = 1$ are

$$|1,1\rangle \quad \text{and} \quad |\tfrac{1}{2};\tfrac{1}{2}\rangle$$

so these must be proportional $|1,1\rangle = e^{i\phi}|\tfrac{1}{2};\tfrac{1}{2}\rangle$

choose $e^{i\phi} = 1$

$$\boxed{|1,1\rangle = |\tfrac{1}{2};\tfrac{1}{2}\rangle}$$

Act on both sides with J_-

$$J_-|1,1\rangle = J_-|\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$= (J_{1-}\otimes 1 + 1\otimes J_{2-})|\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$\hbar\sqrt{2}\,|1,0\rangle = \hbar|-\tfrac{1}{2};\tfrac{1}{2}\rangle + \hbar|\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

$$\boxed{|1,0\rangle = \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle}$$

Act again with J_-

$$J_-|1,0\rangle = (J_{1-}\otimes 1 + 1\otimes J_{2-})\frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2};\tfrac{1}{2}\rangle + |\tfrac{1}{2};-\tfrac{1}{2}\rangle\right)$$

$$\hbar\sqrt{2}\,|1,-1\rangle = \frac{1}{\sqrt{2}}\left(0 + \hbar|-\tfrac{1}{2};-\tfrac{1}{2}\rangle + \hbar|-\tfrac{1}{2};-\tfrac{1}{2}\rangle + 0\right)$$

$$\boxed{|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle}$$

We could now guess

$$|0,0\rangle = \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

using $\langle 0,0|1,0\rangle = 0$ but do it another way

What about $|0,0\rangle$?

We know $J_+|0,0\rangle = 0$

Let $|0,0\rangle = \alpha|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2};\tfrac{1}{2}\rangle$

$$J_+|0,0\rangle = \left(J_{1+}\otimes 1 + 1\otimes J_{2+}\right)\left(\alpha|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2};\tfrac{1}{2}\rangle\right)$$

$$= 0 + \alpha\hbar|\tfrac{1}{2};\tfrac{1}{2}\rangle + \beta\hbar|\tfrac{1}{2};\tfrac{1}{2}\rangle + 0$$

$$= (\alpha+\beta)\hbar\,|\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$\Rightarrow \beta = -\alpha$$

Normalize: $\alpha = \frac{1}{\sqrt{2}}$

singlet $\quad |0,0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2};-\tfrac{1}{2}\rangle - |-\tfrac{1}{2};\tfrac{1}{2}\rangle\right]$

Prob: find CG coeffs for $\underset{\sim}{3}\otimes\underset{\sim}{2} = \underset{\sim}{4}\oplus\underset{\sim}{2}$

Problem

Calc $|\frac{3}{2}, m\rangle$, $|\frac{1}{2}, m\rangle$ in (spin 1) $\otimes$ (spin $\frac{1}{2}$)

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$$
$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle$$

$$|\tfrac{3}{2};\tfrac{3}{2}\rangle = |1;\tfrac{1}{2}\rangle$$
$$\sqrt{3}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{2}\,|0;\tfrac{1}{2}\rangle + |1;-\tfrac{1}{2}\rangle$$
$$2\sqrt{3}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle = 2|-1;\tfrac{1}{2}\rangle + \sqrt{2}\,|0;-\tfrac{1}{2}\rangle + \sqrt{2}\,|0;-\tfrac{1}{2}\rangle$$
$$6\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle = 2|-1,-\tfrac{1}{2}\rangle + 2|-1;-\tfrac{1}{2}\rangle + 2|-1;-\tfrac{1}{2}\rangle$$

$$\begin{cases} |\tfrac{3}{2},\tfrac{3}{2}\rangle = |1;\tfrac{1}{2}\rangle \\ |\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1;-\tfrac{1}{2}\rangle \\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0;-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|-1,\tfrac{1}{2}\rangle \\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle = |-1;-\tfrac{1}{2}\rangle \end{cases}$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \alpha\,|0;+\tfrac{1}{2}\rangle + \beta\,|1;-\tfrac{1}{2}\rangle$$
$$J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = \alpha\sqrt{2}\,|1;+\tfrac{1}{2}\rangle + \beta\,|1;\tfrac{1}{2}\rangle = 0 \quad\Rightarrow\quad \beta = -\sqrt{2}\,\alpha$$

$$\Rightarrow |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle$$
$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|-1;\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle - \tfrac{2}{\sqrt{3}}\,|0;-\tfrac{1}{2}\rangle$$
$$= \sqrt{\tfrac{2}{3}}\,|-1;+\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\,|0;-\tfrac{1}{2}\rangle$$

$$\begin{cases} |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|0;+\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle \\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|-1;\tfrac{1}{2}\rangle \end{cases}$$

$$J_-|1,1\rangle = \sqrt{2}\,|1,0\rangle$$
$$J_-|1,0\rangle = \sqrt{2}\,|1,-1\rangle$$

$$J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

$$J^2\,|1;\tfrac{1}{2}\rangle = 2 + \tfrac{3}{4} + 2(1)(\tfrac{1}{2}) = \tfrac{15}{4} = \tfrac{3}{2}(\tfrac{3}{2}+1) \checkmark$$

$$J^2|\tfrac{3}{2},\tfrac{1}{2}\rangle = \left(\sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1;-\tfrac{1}{2}\rangle\right)$$

$$= (2+\tfrac{3}{4})|\tfrac{3}{2},\tfrac{1}{2}\rangle + \underbrace{\underbrace{\sqrt{\tfrac{2}{3}}\left(\sqrt{2}\,|1;-\tfrac{1}{2}\rangle\right) + \tfrac{1}{\sqrt{3}}\left(2(1)(-\tfrac{1}{2})\,|1;-\tfrac{1}{2}\rangle + \sqrt{2}\,|0;\tfrac{1}{2}\rangle\right)}_{\frac{1}{\sqrt{3}}|1;-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|0;\frac{1}{2}\rangle}}_{|\frac{3}{2},\frac{1}{2}\rangle}$$

$$= \tfrac{15}{4}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle \checkmark$$

$$J^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = J^2\left(\sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle\right)$$

$$= (2+\tfrac{3}{4})\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + \underbrace{\underbrace{\sqrt{\tfrac{1}{3}}\left(\sqrt{2}\,|1;-\tfrac{1}{2}\rangle\right) - \sqrt{\tfrac{2}{3}}\left(2(1)(-\tfrac{1}{2})\,|1;-\tfrac{1}{2}\rangle + \sqrt{2}\,|0;\tfrac{1}{2}\rangle\right)}_{2\sqrt{\frac{2}{3}}|1;-\frac{1}{2}\rangle - 2\frac{1}{\sqrt{3}}|0;\frac{1}{2}\rangle}}_{-2|\frac{1}{2},\frac{1}{2}\rangle}$$

$$= \tfrac{3}{4}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{1}{2}(1+\tfrac{1}{2})\,|\tfrac{1}{2},\tfrac{1}{2}\rangle \checkmark$$

Adding of angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$

A state of the system of two particles belongs to the product space.

$$(\text{spin } j_1) \otimes (\text{spin } j_2)$$

$$\sim (2j_1+1) \otimes (2j_2+1)$$

"spanned" by basis states

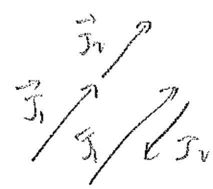
$$|j_1, m_1; j_2, m_2\rangle \quad \text{or} \quad |m_1; m_2\rangle \text{ for short}$$

$m_1 = -j_1, \cdots, j_1$
$m_2 = -j_2, \cdots, j_2$

[Dimension of space $= (2j_1+1)(2j_2+1)$]

One can measure the total angular momentum of a state in this space and obtain various results j ranging from $j_1 + j_2$ to $|j_1 - j_2|$

The space is therefore also a direct sum of representations of total angular momentum



$$\bigoplus_{j=|j_1-j_2|}^{j=j_1+j_2} (\text{spin } j)$$

Spanned by states

$$|j, m\rangle \qquad j = |j_1 - j_2|, \cdots, j_1 + j_2$$
$$m = -j, \cdots, j$$

Example: $j_1 = 1$, $j_2 = 2$

(spin 1) $\otimes$ (spin 2) $=$ (spin 3) + (spin 2) + (spin 1)

$\underset{\sim}{3} \otimes \underset{\sim}{5}$ $\qquad = \underset{\sim}{7} \oplus \underset{\sim}{5} \oplus \underset{\sim}{3}$

Product space basis states $\vert m_1 ; m_2\rangle$			because $J_z = J_{z1} \otimes 1_2 + 1_1 \otimes J_{z2}$ $m = m_1 + m_2$	Direct sum basis states $\vert j, m\rangle$		
			3	$\vert 3,3\rangle$		
$\vert 1;2\rangle$			2	$\vert 3,2\rangle$	$\vert 2,2\rangle$	
$\vert 1;1\rangle$	$\vert 0;2\rangle$		1	$\vert 3,1\rangle$	$\vert 2,1\rangle$	$\vert 1,1\rangle$
$\vert 1;0\rangle$	$\vert 0;1\rangle$	$\vert -1;2\rangle$	0	$\vert 3,0\rangle$	$\vert 2,0\rangle$	$\vert 1,0\rangle$
$\vert 1;-1\rangle$	$\vert 0;0\rangle$	$\vert -1;1\rangle$	-1	$\vert 3,-1\rangle$	$\vert 2,-1\rangle$	$\vert 1,-1\rangle$
$\vert 1;-2\rangle$	$\vert 0;-1\rangle$	$\vert -1;0\rangle$	-2	$\vert 3,-2\rangle$	$\vert 2,-2\rangle$	
	$\vert 0;-2\rangle$	$\vert -1;-1\rangle$	-3	$\vert 3,-3\rangle$		
		$\vert -1;-2\rangle$				
eigenstates of J_{1z} & J_{2z}				eigenstates of J^2, J_z		

The states in any row on the right are linear combinations of states in same row on the left & vice versa

$$|j, m\rangle = \underbrace{\sum_{m_1, m_2}}_{\text{where } m_1 + m_2 = m} C^{j_1 j_2 j}_{m_1 m_2 m} \, |m_1 ; m_2\rangle$$

$\uparrow$ Clebsch-Gordan Coefficients

e.g. $|3,1\rangle = C^{123}_{101} |1;0\rangle + C^{123}_{011} |0;1\rangle + C^{123}_{-1,2,1} |-1,2\rangle$

$\vec{J} = \vec{J}_1 + \vec{J}_2$

$\vec{J}^2 = J_1^2 + 2J_1 \cdot J_2 + J_2^2$

$$\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$$

$$J_i = J_{1i} \otimes 1 + 1 \otimes J_{2i} \quad \text{where } i = x, y, z$$

$$J^2 = \sum_i J_i J_i = \sum (J_{1i} \otimes 1 + 1 \otimes J_{2i})(J_{1i} \otimes 1 + 1 \otimes J_{2i})$$

$$= \sum_i (J_{1i} J_{1i} \otimes 1 + 2 J_{1i} \otimes J_{2i} + 1 \otimes J_{2i} J_{2i})$$

$$= J_1^2 \otimes 1 + 2 (J_{1z} \otimes J_{2z} + J_{1x} \otimes J_{2x} + J_{1y} \otimes J_{2y}) + 1 \otimes J_2^2$$

using $J_{1x} = \frac{1}{2}(J_{1+} + J_{1-})$

$J_{1y} = \frac{1}{2i}(J_{1+} - J_{1-})$

etc

$$\boxed{J^2 = J_1^2 \otimes 1 + 1 \otimes J_2^2 + 2 J_{1z} \otimes J_{2z} + J_{1+} \otimes J_{2-} + J_{1-} \otimes J_{2+}}$$

Note $[J^2, J_{1z} \otimes 1] \neq 0$
$[J^2, 1 \otimes J_{2z}] \neq 0$
so eigenstates of J_{1z} & J_{2z} are not (necessarily) eigenstates of J^2

$$\begin{tikzcd}
 & J_1^2, J_2^2, J_z \arrow[dl, leftrightarrow, "\text{commute}"'] \arrow[dr, leftrightarrow, "\text{commute}"] & \\
J_{1z}, J_{2z} \arrow[rr, leftrightarrow, "\text{don't commute}"'] & & J^2
\end{tikzcd}$$

Consider highest state of product rep

$$J^2 |j_1; j_2\rangle = [\underbrace{j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 + 2 j_1 j_2 \hbar^2}_{(j_1+j_2)(j_1+j_2+1)\hbar^2} + 0 + 0] |j_1; j_2\rangle$$

so $|j_1; j_2\rangle = |j_1 + j_2, j_1 + j_2\rangle$

$$\vec{J} = \vec{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{J}_2$$

$$J^2 = \sum_{i=1}^{3} J_i J_i = J_1^2 \otimes \mathbb{1}_2 + 2\sum_{i=1}^{3} J_{1i} \otimes J_{2i} + \mathbb{1} \otimes J_2^2$$

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$$\cancel{J^2 = (\vec{J}_1 + \vec{J}_2)\cdot(\vec{J}_1 + \vec{J}_2) = J_1^2 + J_2^2 + 2\vec{J}_1\cdot\vec{J}_2}$$

$$= J_1^2 + J_2^2 + 2(J_{1z}J_{2z} + \underbrace{J_{1x}J_{2x}}_{\left(\frac{J_{1+}+J_{1-}}{2}\right)\left(\frac{J_{2+}+J_{2-}}{2}\right)} + \underbrace{J_{1y}J_{2y}}_{\left(\frac{J_{1+}-J_{1-}}{2i}\right)\left(\frac{J_{2+}-J_{2-}}{2i}\right)})$$

$$= J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

(did up to here 12/2013)

[then put this on midterm]

midterm question

Apply this to the triplet states

$$J^2 \left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle = (J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+})\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) + 0 + 0\right)\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$= 2\hbar^2 \left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

so indeed $\left|\tfrac{1}{2};\tfrac{1}{2}\right\rangle = |1,0\rangle$

$$J^2\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle = \underbrace{\left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)\right)}_{\hbar^2}\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + 0 + \hbar^2\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle$$

$$J^2\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle = \hbar^2\left|-\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \hbar^2\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + 0$$

so neither $\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle$ nor $\left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle$ are eigenstates of J^2

$$J^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) = 2\hbar^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) \Rightarrow |1,0\rangle$$

$$J^2\left(\left|\tfrac{1}{2};-\tfrac{1}{2}\right\rangle - \left|-\tfrac{1}{2};\tfrac{1}{2}\right\rangle\right) = 0 \Rightarrow |0,0\rangle$$