$\boxed{\text{Transformation between A-basis and B-basis}}$

[Sakurai, p. 36]

Let $|b_n\rangle$ be basis of eigenstates of operator B. $\left\{ \begin{array}{l} \sum |a_n\rangle\langle a_n| = 1 \\ \sum |b_n\rangle\langle b_n| = 1 \end{array} \right.$

Ket $|\psi\rangle$ represented in B-basis as column vector w/ components

$$\psi_m' = \langle b_m|\psi\rangle = \sum_{k=1}^{d} \langle b_m|a_k\rangle \langle a_k|\psi\rangle = \sum_k \langle b_m|a_k\rangle \psi_k$$

where ψ_k = components in A-basis

Similarly operator C is represented in B-basis as matrix w/

$$C_{mn}' = \langle b_m|C|b_n\rangle = \sum_{k,l} \langle b_m|a_k\rangle \langle a_k|C|a_l\rangle \langle a_l|b_n\rangle$$

$$= \sum \langle b_m|a_k\rangle \underbrace{C_{kl}}_{\text{matrix elements in A-basis}} \langle a_l|b_n\rangle$$

Define transformation operator U w/ matrix elements $U_{ln} \equiv \langle a_l|b_n\rangle$

Hermitian conjugate $U^\dagger$ has elements $(U^\dagger)_{mk} = (U_{km})^* = \langle a_k|b_m\rangle^* = \langle b_m|a_k\rangle$

Then $\psi_m' = \sum_k (U^\dagger)_{mk} \psi_k$

and $C_{mn}' = \sum (U^\dagger)_{mk} C_{kl} U_{ln}$

ie we can transform from A-basis to B-basis by

$$\psi' = U^\dagger \psi$$

$$C' = U^\dagger C U \quad \text{(similarity transformation)}$$

The transformation operator can be constructed as

$$U = \sum_{l,n} |a_l\rangle U_{ln} \langle a_n|$$

$$= \sum_{l,n} |a_l\rangle\langle a_l|b_n\rangle\langle a_n|$$

$$= \underbrace{\left(\sum_l |a_l\rangle\langle a_l|\right)}_{\mathbb{1}} |b_n\rangle\langle a_n|$$

$$= \sum_n |b_n\rangle\langle a_n|$$

(NB. get same result in B-basis; U has same mtx elts in A & B basis)

U converts A-basis element $|a_l\rangle$ into B-basis element $|b_l\rangle$

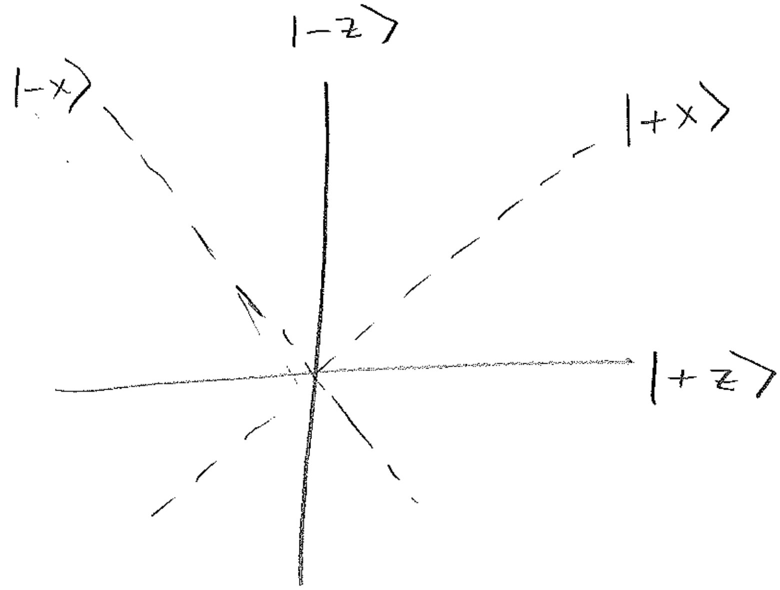
$$U|a_l\rangle = \sum_n |b_n\rangle \underbrace{\langle a_n|a_l\rangle}_{\delta_{nl}} = |b_l\rangle$$

$$U^\dagger = \sum_n |a_n\rangle\langle b_n|$$

U is a <u>unitary</u> matrix which means $U^\dagger U = \mathbb{1}$

Check: $U^\dagger U = \left(\sum_n |a_n\rangle\langle b_n|\right)\left(\sum_l |b_l\rangle\langle a_l|\right)$

$$= \sum_{n,l} |a_n\rangle \underbrace{\langle b_n|b_l\rangle}_{\delta_{nl}} \langle a_l| = \sum |a_n\rangle\langle a_n| = \mathbb{1}$$



$$|+x\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\left(-|+\rangle + |-\rangle\right)$$

← picture only possible because coefficients are real

<u>Observe</u>

$|\pm z\rangle$ are orthogonal

$|\pm x\rangle$ are orthogonal

$|\pm z\rangle$ + $|\pm x\rangle$ related by a rotation in state space

more generally a unitary transformation

Let U = unitary transformation from S_z-basis to S_x-basis

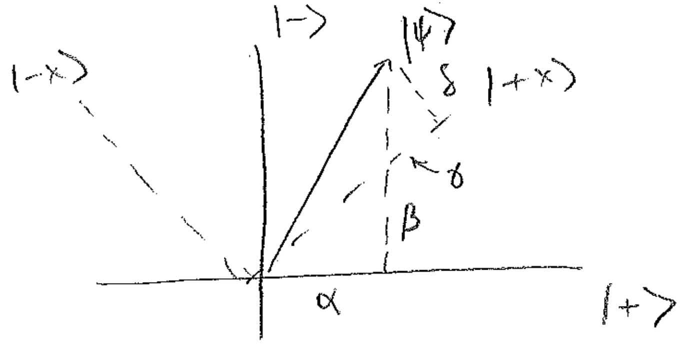
$$U = \sum |b_n\rangle\langle a_n| = |+x\rangle\langle +| + |-x\rangle\langle -|$$

U has matrix element $U_{\ell n} = \langle a_\ell | b_n\rangle$

$$U = \begin{pmatrix} \langle + | +x\rangle & \langle + | -x\rangle \\ \langle - | +x\rangle & \langle - | -x\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

check $U^\dagger U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



Let $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \gamma|+x\rangle + \delta|-x\rangle$

components in S_z basis

components in S_x basis

The components in S_z + S_x basis are related by

$$\psi' = U^{\dagger}\psi$$

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{2}}(\alpha+\beta)$$

$$\delta = \frac{1}{\sqrt{2}}(-\alpha+\beta)$$

But $U^{\dagger} = U^{-1}$ so we also have $\psi' = U^{-1}\psi$

$$\Rightarrow \psi = U\psi'$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix}$$

S_x in S_z-basis is $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

What is S_x in S_x-basis?

$$S_x' = U^\dagger S_x U$$

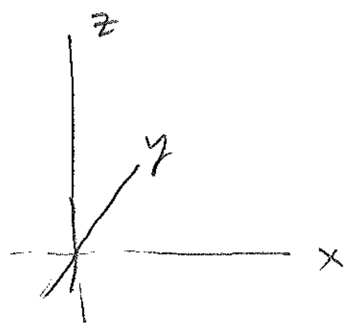
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \checkmark$$

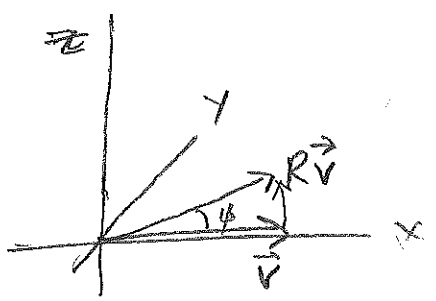
Rotations in physical space

Vectors in <u>physical</u> space are rotated by orthogonal 3×3 rotation matrices R

Orthogonal means $R^T R = \mathbb{1}$



Consider a rotation through ϕ counterclockwise about $\hat{z}$ axis (thumb in $\hat{z}$ direction, fingers of right hand curl in direction of rotation)



$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(NB opposite Phys 2140 because active not passive)

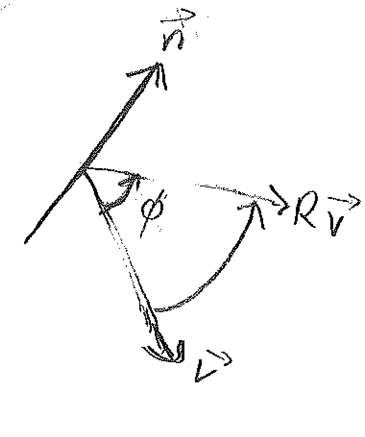
$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R\vec{v} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

[think about signs!]

More generally, consider a rotation about some axis $\vec{n}$ through an angle ϕ.

(where $\vec{n}\cdot\vec{n} = 1$)

(don't use $\hat{n}$: operator)



Unitary transformations in state space

A rotation in physical space Λ is implemented by a unitary transformation U on kets in space state (Hilbert space)

Unitary means $U^\dagger U = \mathbb{1}$

claim: A rotation about axis $\vec{n}$ thru ϕ Λ is implemented via

$$U = e^{-\frac{i\phi \vec{n}\cdot\vec{J}}{\hbar}}$$

where $\vec{J}$ = angular momentum operator.

We say angular momentum is the "generator of rotations"

In C.M. invariance of laws of physics under rotation implies existence of a conserved quantity, viz ang. mom. (Noether's thm)

In QM, the ang. mom. operator generates rotations via U

Suppose we are interested in the state of a spin-$\frac{1}{2}$ particle.

Then $\vec{J} = \vec{S}$ = spin operator.

In the S_z-basis, $\vec{S}$ is represented by Pauli spin matrices $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ so

$$U = e^{-\frac{i\phi\,\vec{n}\cdot\vec{\sigma}}{2}} = e^{-\frac{i\phi}{2}(\vec{n}\cdot\vec{\sigma})}$$

$$= \mathbb{1} - \frac{i\phi}{2}\vec{n}\cdot\vec{\sigma} + \frac{1}{2}\left(-\frac{i\phi}{2}\right)^2(\vec{n}\cdot\vec{\sigma})^2 + \frac{1}{3!}\left(-\frac{i\phi}{2}\right)^3(\vec{n}\cdot\vec{\sigma})^3 + \dots$$

[HW] Now $(\vec{n}\cdot\vec{\sigma})^2 = \mathbb{1}$ and $(\vec{n}\cdot\vec{\sigma})^3 = \vec{n}\cdot\vec{\sigma}$ etc so

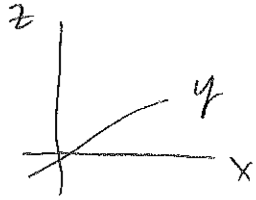
$$U = \left(1 - \frac{1}{2}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 + \dots\right)\mathbb{1} - i\left(\frac{\phi}{2} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \dots\right)\vec{n}\cdot\vec{\sigma}$$

$$\boxed{U = \boxed{e^{-\frac{i\phi}{2}(\vec{n}\cdot\vec{\sigma})} = \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)\vec{n}\cdot\vec{\sigma}}}$$ for spin-$\frac{1}{2}$ $\leftarrow$ (matrix version of Euler thm)

$$U^\dagger = \cos\left(\frac{\phi}{2}\right)\mathbb{1} + i\sin\left(\frac{\phi}{2}\right)\hat{n}\cdot\vec{\sigma}$$

$$U^\dagger U = \cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right)(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1} \quad \checkmark$$

$$U = \begin{pmatrix} \cos\frac{\phi}{2} - i n_z \sin\frac{\phi}{2} & (-i n_x - n_y)\sin\frac{\phi}{2} \\ (-i n_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + i n_z \sin\frac{\phi}{2} \end{pmatrix}$$



If we want $\hat{z}$-axis $\longrightarrow$ $\hat{x}$-axis

rotate about $\hat{n} = \hat{y}$ by $\phi = \frac{\pi}{2}$

This is implemented via

$$U = \underbrace{\cos\left(\frac{\pi}{4}\right)}_{1/\sqrt{2}} \mathbb{1} - i \underbrace{\sin\left(\frac{\pi}{4}\right)}_{1/\sqrt{2}} \sigma_y \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$ agrees with what we found earlier

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Note: consider rotation about any axis by 2π radians

$$U = (\underbrace{\cos \pi}_{-1}) \mathbb{1} - i \underbrace{\sin(\pi)}_{0} \; \hat{\omega} \cdot \vec{\sigma}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi' = U^{+} \psi = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

↑ change sign

(unobservable unless do interference experiment) → discuss later after electron precession

[Sakurai]