Matrix representations in the $\hat{A}$-basis

Given an orthonormal basis (e.g. d nondegenerate eigenstates $|a_n\rangle$ of Hermitian operator $\hat{A}$)

any operator $\hat{B}$ can be represented as a $d \times d$ matrix whose matrix elements in the A-basis are

$$B_{mn} = \langle a_m | B | a_n \rangle$$

For example, A in the A-basis is represented by

$$A_{mn} = \langle a_m | A | a_n \rangle = a_m \langle a_m | a_n \rangle = a_n \delta_{mn}$$

use orthonormality

$$A \to \begin{pmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_d \end{pmatrix},$$

a diagonal matrix

The product of 2 operators is represented by the matrix product

use completeness

$$\langle a_m | BC | a_n \rangle = \langle a_m | B \mathbb{1} C | a_n \rangle = \sum_l \langle a_m | B | a_l \rangle \langle a_l | C | a_n \rangle$$

$$= \sum_l B_{ml} C_{ln}$$

Reconstruct operator from matrix element

$$B_{mn} \to B = \sum_{m,n=1}^{d} |a_m\rangle B_{mn} \langle a_n|$$

A ket $|\psi\rangle$ is represented in the A-basis as a d-dim'l column vector with entries $\psi_n = \langle a_n|\psi\rangle$

$$|\psi\rangle \rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix} = \begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \\ \vdots \\ \langle a_d|\psi\rangle \end{pmatrix}$$

The basis kets are

$$|a_n\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow n^{\text{th}} \text{ entry}$$

Reconstruct ket from components $\psi_n \rightarrow |\psi\rangle = \sum_{n=1}^{d} \psi_n |a_n\rangle$

A bra $\langle\psi|$ is represented in the A-basis as a d-dim'l row vector with entries $\psi_n^* = \langle\psi|a_n\rangle$

$$\langle\psi| \rightarrow (\psi_1^*, \psi_2^*, \ldots \psi_d^*) = (\langle\psi|a_1\rangle, \ldots, \langle\psi|a_d\rangle)$$

$$\langle a_n| \rightarrow (0, 0, \ldots \underset{\uparrow\, n^{\text{th}} \text{ entry}}{1}, \ldots 0)$$

Reconstruct

$$\langle\psi| = \sum \psi_n^* \langle a_n|$$

Inner product

$$\langle \phi | \psi \rangle = \sum_{n=1}^{d} \langle \phi | a_n \rangle \langle a_n | \psi \rangle = \sum \langle a_n | \phi \rangle^* \langle a_n | \psi \rangle = \sum \phi_n^* \psi_n$$

is just the product of row + column vectors

$$(\phi_1^*, \dots \phi_d^*) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

Eigenvalue equation (possibly omit)

$$B | \psi \rangle = \lambda | \psi \rangle$$

$$\sum_{m=1}^{d} \langle a_n | B | a_m \rangle \langle a_m | \psi \rangle = \lambda \langle a_n | \psi \rangle$$

$$\sum_m B_{nm} \psi_m = \lambda \psi_n$$

matrix eigenvalue equation

[added after 2017] A projection operator $|a_n\rangle\langle a_n|$ is represented by a matrix

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{d \times 1} \underbrace{(0 \; 0 \dots 1 \dots 0)}_{1 \times d} = \begin{pmatrix} 0 & & & \\ & 0 & & 0 \\ & & \ddots 1 & \\ 0 & & & \ddots 0 \end{pmatrix}$$

observe $\sum |a_n\rangle\langle a_n| = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \underset{\text{unit mtx}}{\mathbb{1}}$

Hermitian conjugate of an operator → transpose complex conj of matrix

$$|\chi\rangle = B|\psi\rangle \quad \Rightarrow \quad \langle\chi| = \langle\psi| B^\dagger$$

$$\langle a_n|\chi\rangle = \sum_m \langle a_n|B|a_m\rangle\langle a_m|\psi\rangle \qquad \langle\chi|a_n\rangle = \sum_m \langle\psi|a_m\rangle\langle a_m|B^\dagger|a_n\rangle$$

$$\chi_n = \sum_m B_{nm}\psi_m \qquad \chi_n^* = \sum_m \psi_m^* (B^\dagger)_{mn}$$

[matrix times col. vector]

$$\chi_n^* = \sum B^*_{nm}\psi^*_m$$

$$= \sum \psi^*_m (B^{*T})_{mn}$$

so $\quad (B^\dagger)_{mn} = (B^{*T})_{mn}$

<u>Spin-$\frac{1}{2}$ system</u>: matrix representation in the S_z basis

$|+\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|-\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \longrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ← called a "spinor"

$$B \longrightarrow \begin{pmatrix} \langle +|B|+\rangle & \langle +|B|-\rangle \\ \langle -|B|+\rangle & \langle -|B|-\rangle \end{pmatrix}$$

$$S_z \longrightarrow \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$ [diagonal as expected w/ eigenvalues]

Recall $S_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$

$$S_x \longrightarrow \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$S_y \longrightarrow \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}$$

[exercise: show $[S_x, S_y] = i\hbar S_z$
also $S^2 = S_x^2 + S_y^2 + S_z^2$
as mtx eqns]

$$S^2 \longrightarrow \begin{pmatrix} \frac{3\hbar^2}{4} & 0 \\ 0 & 3\hbar^2/4 \end{pmatrix}$$

Define $S_i = \frac{\hbar}{2}\sigma_i$ $\qquad \sigma_i$ = Pauli spin matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Observe $\sigma_i^\dagger = (\sigma_i^T)^* = \sigma_i$ Hermitian matrices
(because S_i is an observable)

exercise

(a)

$$[S_x, S_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}\right] = i\hbar S_z$$

$$[S_y, S_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} & 2i \\ 2i & \end{pmatrix}\right] = i\hbar S_x$$

$$[S_z, S_x] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\hbar S_y$$

(b)

$$\frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right] = \frac{3\hbar^2}{4}\mathbb{1} \checkmark$$

Eigenvectors of S_z

We already know that

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

operator → matrix

$$S_z \rightarrow \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

ket → col. vector

$$|+\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$$|-\rangle \Rightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

check: $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$ ✓

$|\pm\rangle$ are also eigenstates of S^2 w/ eigenvalue $\frac{3\hbar^2}{4}$

$$S^2 \longrightarrow \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Complete set of simultaneous eigenstates
$\Rightarrow$ S^2 and S_z compatible

$[S^2, S_z] = 0$ because identity commutes w/ anything

Eigenvectors of S_x

$$S_x |\psi\rangle = \lambda |\psi\rangle$$

Solve using matrix rep. in S_z-basis:

$$S_x \rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$|\psi\rangle \rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2}\sigma_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left(\frac{\hbar}{2}\sigma_x - \lambda \mathbb{1}\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Trivial solution $(\alpha = \beta = 0)$ unless $\det = 0$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \implies \lambda = \pm\frac{\hbar}{2}$$

no surprise

Let $\lambda = \frac{\hbar}{2}$

$$\begin{pmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} -\alpha+\beta \\ \alpha-\beta \end{pmatrix} = 0 \quad \Rightarrow \quad \alpha = \beta$$

$$|\psi\rangle \Rightarrow \begin{pmatrix} \beta \\ \beta \end{pmatrix}$$

Normalize: $\langle\psi|\psi\rangle = (\beta^* \ \beta^*)\begin{pmatrix} \beta \\ \beta \end{pmatrix} = 2|\beta|^2 = 1 \quad \Rightarrow \quad \beta = \frac{1}{\sqrt{2}} e^{i\theta}$

Overall phase arbitrary: choose $\theta = 0$

$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ call this $|+x\rangle$

$$|+x\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Let $\lambda = -\frac{\hbar}{2}$

$$\begin{pmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & \frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} \alpha+\beta \\ \alpha+\beta \end{pmatrix} = 0 \quad \Rightarrow \quad \alpha = -\beta$$

$$|\psi\rangle \rightarrow \begin{pmatrix} -\beta \\ \beta \end{pmatrix}$$

Normalize $\Rightarrow$ $\beta = \frac{1}{\sqrt{2}} e^{i\theta}$ & choose $\theta = 0$

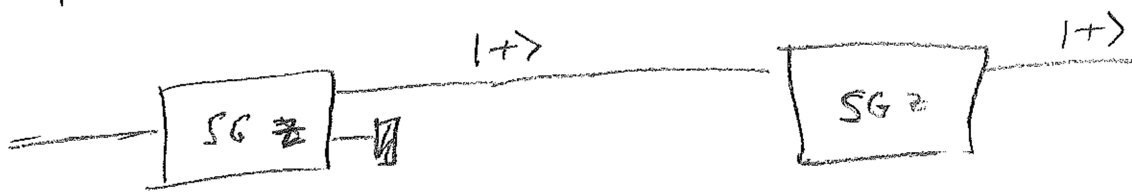
$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ call this $|-x\rangle$

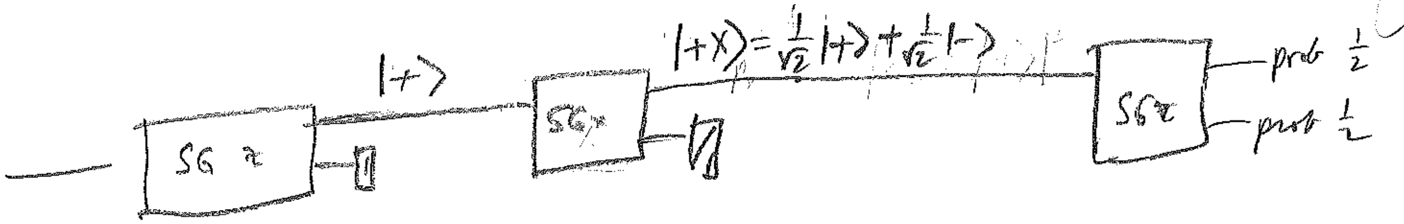
$$|-x\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

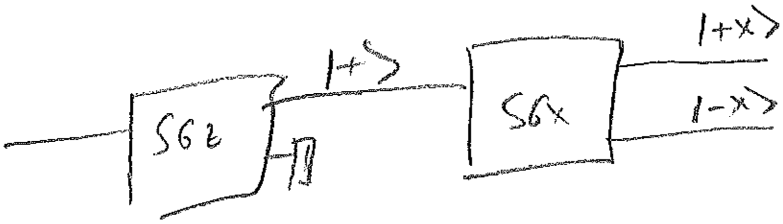
- Because $[S_x, S_z] \neq 0$, S_x + S_z are incompatible and so eigenstates $|\pm x\rangle$ are distinct from $|\pm z\rangle$

- Because $[S^2, S_x] = 0$ (since S^2 prop to identity) S^2 + S_x are compatible and so $|\pm x\rangle$ are also S^2 eigenstates

Return to Stern Gerlach







← prob: $|\langle +x|+\rangle|^2 = \frac{1}{2}$

← prob $|\langle -x|+\rangle|^2 = \frac{1}{2}$

$$\langle +x|+\rangle = \frac{1}{\sqrt{2}}(1, 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

or

$$\langle +x| = \frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|$$